Outline

1. Lecture 1: gravitational wave astronomy, the two-body problem, and self-force theory
2. Lecture 2: the local problem: how to deal with small bodies
    - Perturbation theory in GR
    - Small bodies and punctures
    - Point particles and mode-sum regularization
3. Lecture 3: the global problem: orbital dynamics in Kerr
4. Lecture 4: the global problem: black hole perturbation theory

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Perturbative Einstein equations

If the exact metric is $\hat{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$, then

$$C^{\alpha}_{\beta\gamma} := \hat{\Gamma}^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\hat{g}^{\alpha\mu}(2\nabla_{(\beta}h_{\gamma)\mu} - \nabla_{\mu}h_{\beta\gamma})$$

$$\Rightarrow \hat{R}^{\alpha}{}_{\beta\gamma\delta}v^{\beta} = (\hat{\nabla}_{\gamma}\hat{\nabla}_{\delta} - \hat{\nabla}_{\delta}\hat{\nabla}_{\gamma})v^{\alpha} = \left(R^{\alpha}{}_{\beta\gamma\delta} + 2\nabla_{[\gamma}C^{\alpha}_{\delta]\beta} + 2C^{\alpha}_{\mu[\gamma}C^{\mu}_{\delta]\beta}\right)v^{\beta}$$

$$\Rightarrow \hat{R}_{\beta\delta} = R_{\beta\delta} + 2\nabla_{[\alpha}C^{\alpha}_{\delta]\beta} + 2C^{\alpha}_{\mu[\alpha}C^{\mu}_{\delta]\beta}$$

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Perturbative Einstein equations continued

- expand in powers of nonlinearity: $\hat{g}^{\alpha\beta} = g^{\alpha\beta} - h^{\alpha\beta} + \frac{1}{2}h^{\alpha}{}_{\gamma}h^{\gamma\beta} + \dots$

$$\Rightarrow \hat{R}_{\alpha\beta} = R_{\alpha\beta} + R^{(1)}_{\alpha\beta}[h] + R^{(2)}_{\alpha\beta}[h,h] + \dots$$

- linearized Ricci tensor:

$$R^{(1)}_{\alpha\beta}[h] = -\frac{1}{2}\nabla^{\mu}\nabla_{\mu}h_{\alpha\beta} - \frac{1}{2}\nabla_{\alpha}\nabla_{\beta}(g^{\mu\nu}h_{\mu\nu}) + \nabla^{\mu}\nabla_{(\alpha}h_{\beta)\mu}$$

$$= -\frac{1}{2}\left(\nabla^{\mu}\nabla_{\mu}h_{\alpha\beta} + 2R_{\alpha}{}^{\mu}{}_{\beta}{}^{\nu}h_{\mu\nu}\right) + \nabla_{(\alpha}\nabla^{\mu}\bar{h}_{\beta)\mu}$$

(trace-reversed perturbation: $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}h_{\mu\nu}$)

- quadratic piece of Ricci tensor:

$$R^{(2)}_{\alpha\beta}[h,h] \sim \nabla h \nabla h + h\nabla\nabla h$$

Perturbative Einstein equations continued

- now consider one-parameter family of spacetimes with metric $\hat{g}_{\alpha\beta}(\epsilon) = g_{\alpha\beta} + h_{\alpha\beta}(\epsilon)$ and stress-energy $\hat{T}_{\alpha\beta}(\epsilon)$
- substitute $h_{\alpha\beta} = \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + O(\epsilon^3)$

$$\Rightarrow \hat{R}_{\alpha\beta} = R_{\alpha\beta} + \epsilon R^{(1)}_{\alpha\beta}[h^{(1)}] + \epsilon^2 \left(R^{(1)}_{\alpha\beta}[h^{(2)}] + R^{(2)}_{\alpha\beta}[h^{(1)}, h^{(1)}] \right) + O(\epsilon^3)$$

- substitute $\hat{T}_{\alpha\beta}(\epsilon) = T_{\alpha\beta} + \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta} + O(\epsilon^3)$

$$\Rightarrow \qquad \begin{aligned} G_{\alpha\beta} &= 8\pi T_{\alpha\beta}, \\ G^{(1)}_{\alpha\beta}[h^{(1)}] &= 8\pi T^{(1)}_{\alpha\beta}, \\ G^{(1)}_{\alpha\beta}[h^{(2)}] &= 8\pi T^{(2)}_{\alpha\beta} - G^{(2)}_{\alpha\beta}[h^{(1)}, h^{(1)}], \\ &\;\;\vdots \end{aligned}$$

Gauge freedom: infinitesimal coordinate transformations

Make a small coordinate transformation:

$$x^\mu \to x'^\mu = x^\mu - \epsilon\xi^\mu + O(\epsilon^2)$$

Expand the metric in the two coordinate systems:

$$\hat{g}_{\mu\nu}(x,\epsilon) = g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x) + O(\epsilon^2)$$
$$\hat{g}'_{\mu\nu}(x',\epsilon) = g_{\mu\nu}(x') + \epsilon h'^{(1)}_{\mu\nu}(x') + O(\epsilon^2)$$

How are they related? Tensor transformation law:

$$\hat{g}'_{\mu\nu}(x',\epsilon) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\hat{g}_{\alpha\beta}(x(x'),\epsilon)$$

Expand $x^\mu(x'^\nu)$ and $\hat{g}_{\alpha\beta}$:

$$\hat{g}'_{\mu\nu}(x') = g_{\mu\nu}(x') + \epsilon[h^{(1)}_{\mu\nu}(x') + \mathcal{L}_\xi g_{\mu\nu}(x')] + O(\epsilon^2)$$

So we find

$$h'^{(1)}_{\mu\nu} = h^{(1)}_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$$

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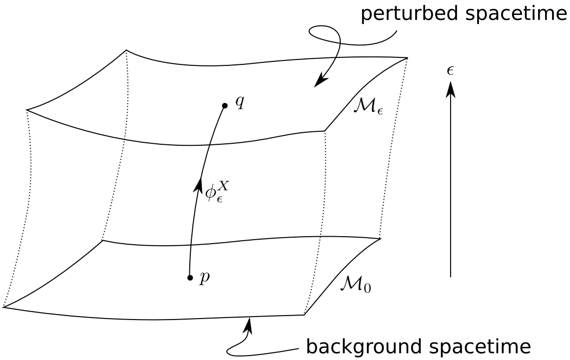
Gauge freedom: geometrical description

- expansion in powers of ϵ is expansion along flow lines through the family:

$$(\phi_\epsilon^{X*}\hat{g})_{\mu\nu}(p) = \hat{g}_{\mu\nu}(p) + \epsilon\mathcal{L}_X\hat{g}_{\mu\nu}(p) + \tfrac{1}{2}\epsilon^2\mathcal{L}_X^2\hat{g}_{\mu\nu}(p) + O(\epsilon^3)$$



- $h^{(1)}_{\mu\nu} = \mathcal{L}_X\hat{g}_{\mu\nu}|_{\epsilon=0}$ and $h'^{(1)}_{\mu\nu} = \mathcal{L}_Y\hat{g}_{\mu\nu}|_{\epsilon=0}$

$$\Rightarrow \Delta h^{(1)}_{\mu\nu}(p) = \mathcal{L}_Y\hat{g}_{\mu\nu}(p) - \mathcal{L}_X\hat{g}_{\mu\nu}(p) = \mathcal{L}_{Y-X}\hat{g}_{\mu\nu}(p) = \mathcal{L}_\xi g_{\mu\nu}(p)$$

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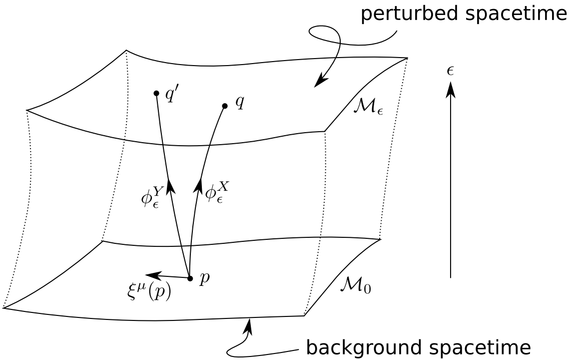

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Lorenz gauge

- define $\bar{h}_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}h_{\mu\nu}$
- gauge condition $\nabla_\beta \bar{h}^{\alpha\beta} = 0$

$$\Rightarrow R^{(1)}_{\alpha\beta}[h] = -\frac{1}{2}\left(\nabla^\mu\nabla_\mu h_{\alpha\beta} + 2R_\alpha{}^\mu{}_\beta{}^\nu h_{\mu\nu}\right)$$

$$G^{(1)}_{\alpha\beta}[h] = -\frac{1}{2}\left(\nabla^\mu\nabla_\mu \bar{h}_{\alpha\beta} + 2R_\alpha{}^\mu{}_\beta{}^\nu \bar{h}_{\mu\nu}\right)$$

- commonly used in self-force theory

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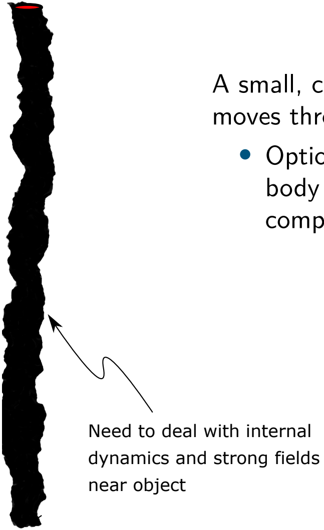
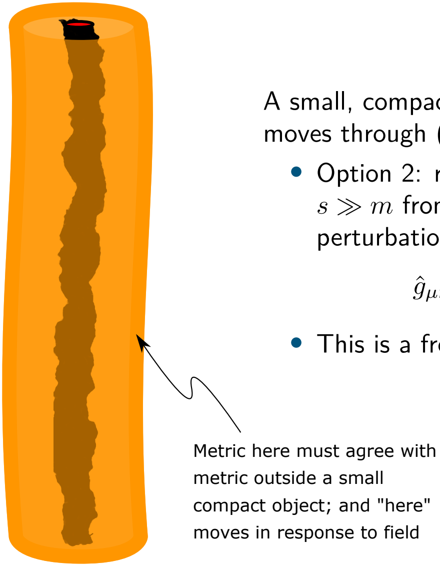
What is the problem we want to solve?



A small, compact object of mass and size $m \sim l \sim \epsilon$ moves through (and influences) spacetime

- Option 1: tackle the problem directly, treat the body as finite sized, deal with its internal composition

What is the problem we want to solve?



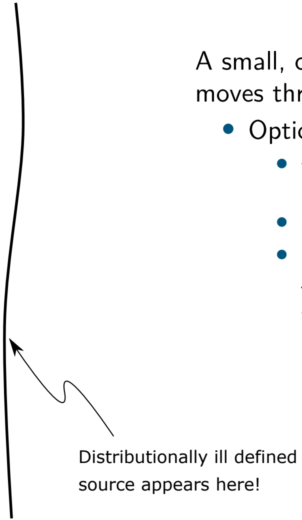
A small, compact object of mass and size $m \sim l \sim \epsilon$ moves through (and influences) spacetime

- Option 2: restrict the problem to distances $s \gg m$ from the object, treat m as source of perturbation of external background $g_{\mu\nu}$:

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \ldots$$

- This is a free boundary value problem

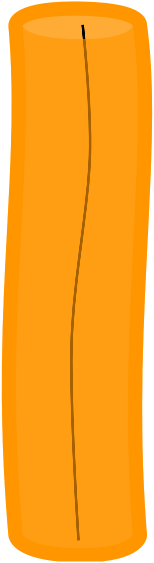
What is the problem we want to solve?



A small, compact object of mass and size $m \sim l \sim \epsilon$ moves through (and influences) spacetime

- Option 3: treat the body as a point particle
    - takes behavior of fields outside object and extends it down to a fictitious worldline
    - so $h^{(1)}_{\mu\nu} \sim 1/s$ ($s =$distance from object)
    - $G^{(1)}_{\mu\nu}[h^{(2)}] \sim G^{(2)}_{\mu\nu}[h^{(1)}] \sim (\nabla h^{(1)})^2 \sim 1/s^4$ —no solution unless we restrict it to points off worldline, which is equivalent to FBVP

What is the problem we want to solve?

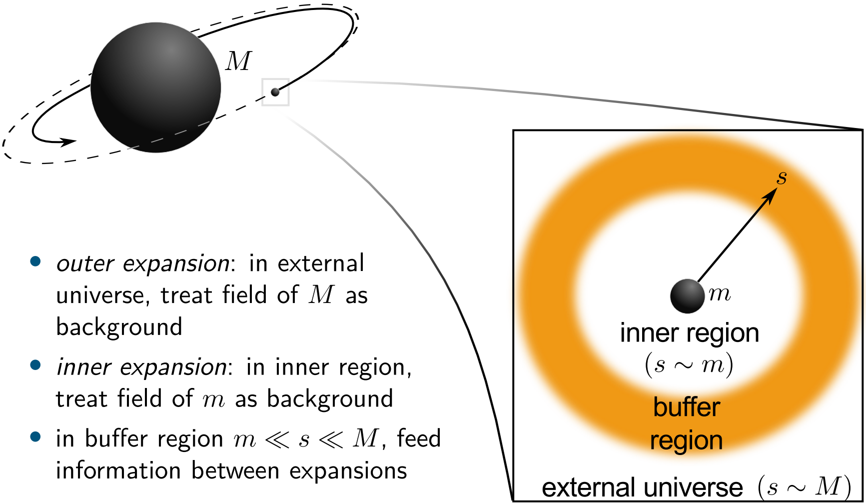


A small, compact object of mass and size $m \sim l \sim \epsilon$ moves through (and influences) spacetime

- Option 4: transform the FBVP into an *effective* problem using a *puncture*, a local approximation to the field outside the object
- this will be the method emphasized here

[Mino, Sasaki, Tanaka 1996; Quinn & Wald 1996; Detweiler & Whiting 2002-03; Gralla & Wald 2008-2012; Pound 2009-2017; Harte 2012]

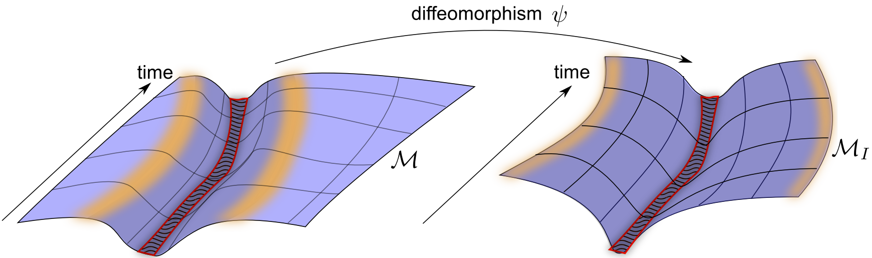
Matched asymptotic expansions



- *outer expansion*: in external universe, treat field of M as background
- *inner expansion*: in inner region, treat field of m as background
- in buffer region $m \ll s \ll M$, feed information between expansions

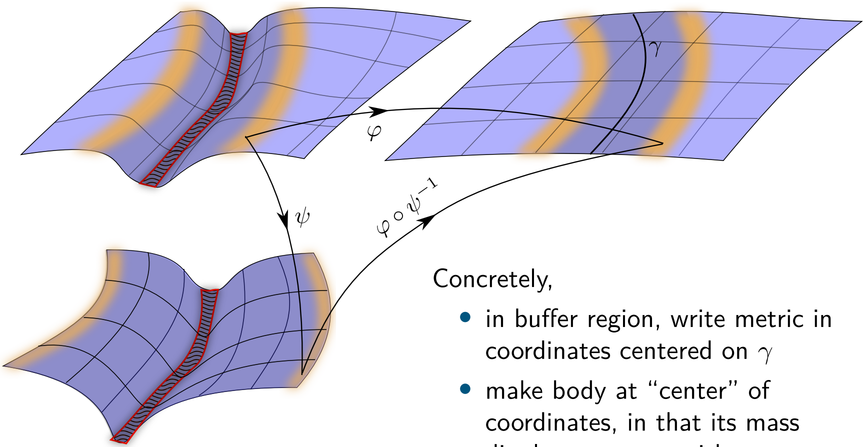
Inner expansion: zoom in on body

- use scaled coords $\tilde{s} \sim s/\epsilon$ to keep size of body fixed, send other distances to infinity as $\epsilon \to 0$
- unperturbed body defines background spacetime $g^{\rm body}_{\mu\nu}$ in inner expansion
- buffer region at asymptotic infinity $s \gg m$
  $\Rightarrow$ can define multipole moments without integrals over body



Effective worldline

- Effective worldline γ in external spacetime defined by body's "centredness" in body's spacetime



Concretely,

- in buffer region, write metric in coordinates centered on γ
- make body at "center" of coordinates, in that its mass dipole moment vanishes

Matching condition

- outer: $\hat{g}_{\mu\nu}(s,\epsilon) = g_{\mu\nu}(s) + \epsilon h^{(1)}_{\mu\nu}(s) + \epsilon^2 h^{(2)}_{\mu\nu}(s) + O(\epsilon^3)$
- inner: $\hat{g}_{\mu\nu}(s/\epsilon,\epsilon) = g^{\rm body}_{\mu\nu}(s/\epsilon) + \epsilon H^{(1)}_{\mu\nu}(s/\epsilon) + \epsilon^2 H^{(2)}_{\mu\nu}(s/\epsilon) + O(\epsilon^3)$
- matching condition:
    - expand outer expansion for small s:

$$\hat{g}_{\mu\nu} = \sum_{n\geq 0}\sum_{p} \epsilon^n s^p \hat{g}^{(n,p)}_{\mu\nu}$$

    - expand inner expansion for small ϵ:

$$\hat{g}_{\mu\nu} = \sum_{n\geq 0}\sum_{p} \epsilon^n (\epsilon/s)^p \check{g}^{(n,p)}_{\mu\nu}$$

    - they must agree:

$$\hat{g}^{(n,p)}_{\mu\nu} = \check{g}^{(n+p,-p)}_{\mu\nu}$$

Form of metric in buffer region

- matching conditions constrains dependence on s:

  e.g., inner expansion must not have negative powers of ϵ

  $\Rightarrow$ most singular power of s in $\epsilon^n h^{(n)}_{\mu\nu}(s)$ is $\dfrac{\epsilon^n}{s^n} = \dfrac{\epsilon^n}{\epsilon^n \tilde{s}^n} = \dfrac{1}{\tilde{s}^n}$

  $$\Rightarrow h^{(n)}_{\mu\nu} = \frac{1}{s^n} h^{(n,-n)}_{\mu\nu} + s^{-n+1} h^{(n,-n+1)}_{\mu\nu} + s^{-n+2} h^{(n,-n+2)}_{\mu\nu} + \dots$$

- $h^{(n,-n)}_{\mu\nu}/\tilde{s}^n$ must equal a term in asymptotic expansion $g^{\rm body}_{\mu\nu}(\tilde{s})$

  $\Rightarrow h^{(n,-n)}_{\mu\nu}$ is determined by multipole moments of isolated body

Form of metric in buffer region

Solving the field equations:

- substitute expansion of $h^{(n)}_{\mu\nu}$ into field equations, solve order by order in s
- expand each $h^{(n,p)}_{\mu\nu}$ in spherical harmonics
- given a worldline γ, the solution at all orders is fully characterized by
  1. body's multipole moments (and corrections thereto): $\sim \frac{Y^{\ell m}}{s^{\ell+1}}$
  2. smooth solutions to vacuum wave equation: $\sim s^{\ell} Y^{\ell m}$
- everything else made of (linear or nonlinear) combinations of the above

Self field and regular field

- multipole moments define $h^{\mathrm{S}(n)}_{\mu\nu}$; interpret as bound field of body
- smooth homogeneous solutions define $h^{\mathrm{R}(n)}_{\mu\nu}$; free radiation, determined by global boundary conditions

General solution in buffer region

First order

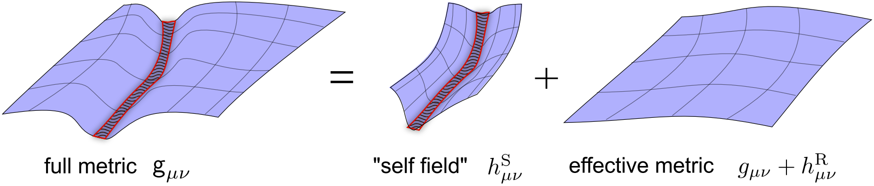
- $h^{(1)}_{\mu\nu} = h^{\mathrm{S}(1)}_{\mu\nu} + h^{\mathrm{R}(1)}_{\mu\nu}$
- $h^{\mathrm{S}(1)}_{\mu\nu} \sim \dfrac{m}{s} + O(s^0)$ defined by mass monopole m
- $h^{\mathrm{R}(1)}_{\mu\nu}$ is undetermined homogenous solution regular at $s = 0$

Second order [Pound 2009, 2012, Gralla 2012]

- $h^{(2)}_{\mu\nu} = h^{\mathrm{S}(2)}_{\mu\nu} + h^{\mathrm{R}(2)}_{\mu\nu}$
- $h^{\mathrm{S}(2)}_{\mu\nu} \sim \dfrac{m^2 + S^i}{s^2} + \dfrac{\delta m + m h^{\mathrm{R}(1)}}{s} + O(s^0)$ defined by

  1. monopole correction δm
  2. spin dipole S^i
  3. terms $\propto m h^{\mathrm{R}(1)}_{\mu\nu}$

Self-field and effective field



- $h^{\rm S}_{\mu\nu}$ directly determined by object's multipole moments
- $g_{\mu\nu}+h^{\rm R}_{\mu\nu}$ is a *smooth vacuum metric* determined by global boundary conditions

Equations of motion

Solving EFE in buffer region yields equations of motion for object's effective center of mass

1st order, arbitrary compact object [MiSaTaQuWa 1996]:

$$\frac{D^2 z^\mu}{d\tau^2} = -\tfrac{1}{2}\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{\mathrm{R1}}_{\delta\beta;\gamma} - h^{\mathrm{R1}}_{\beta\gamma;\delta}\right)u^\beta u^\gamma + \tfrac{1}{2m} R^\alpha{}_{\beta\gamma\delta} u^\beta S^{\gamma\delta} + O(m^2)$$

(motion of spinning test body in $g_{\mu\nu} + h^{\mathrm{R1}}_{\mu\nu}$)

2nd-order, nonspinning, spherical compact object [Pound 2012]:

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(geodesic motion in $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$)

- these results are derived *directly from EFE outside the object*; there's no regularization of infinities, and no assumptions about $h^{\mathrm{R}}_{\mu\nu}$

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$$\frac{D^2 z^\mu}{d\tau^2} = -\tfrac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(g_\nu{}^\rho - h^{\rm R}_\nu{}^\rho\right)\left(2h^{\rm R}_{\rho\sigma;\lambda} - h^{\rm R}_{\sigma\lambda;\rho}\right)u^\sigma u^\lambda + O(m^3)$$

(geodesic motion in $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{\rm R}_{\mu\nu}$)

- these results are derived *directly from EFE outside the object*; there's no regularization of infinities, and no assumptions about $h^{\rm R}_{\mu\nu}$

Equations of motion

Solving EFE in buffer region yields equations of motion for object's effective center of mass

1st order, arbitrary compact object [MiSaTaQuWa 1996]:

$$\frac{D^2 z^\mu}{d\tau^2} = -\tfrac{1}{2}\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(2h^{\mathrm{R1}}_{\delta\beta;\gamma} - h^{\mathrm{R1}}_{\beta\gamma;\delta}\right)u^\beta u^\gamma + \tfrac{1}{2m} R^\alpha{}_{\beta\gamma\delta} u^\beta S^{\gamma\delta} + O(m^2)$$

(motion of spinning test body in $g_{\mu\nu} + h^{\mathrm{R1}}_{\mu\nu}$)

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$$\frac{D^2 z^\mu}{d\tau^2} = -\tfrac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(g_\nu{}^\rho - h^{\mathrm{R}\,\rho}_\nu\right)\left(2h^{\mathrm{R}}_{\rho\sigma;\lambda} - h^{\mathrm{R}}_{\sigma\lambda;\rho}\right)u^\sigma u^\lambda + O(m^3)$$

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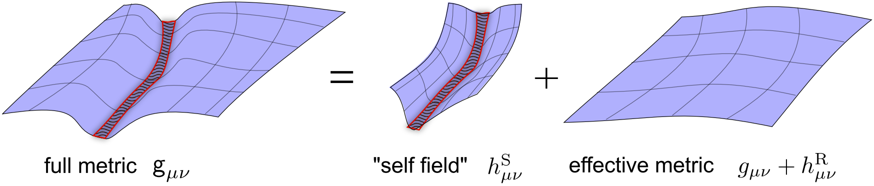
Punctures [Barack et al, Detweiler, Gralla-Wald, Pound]

- replace "self-field" with "singular field"



- replace object with a *puncture*, a local singularity in the field, moving on z^{μ}, equipped with the object's multipole moments

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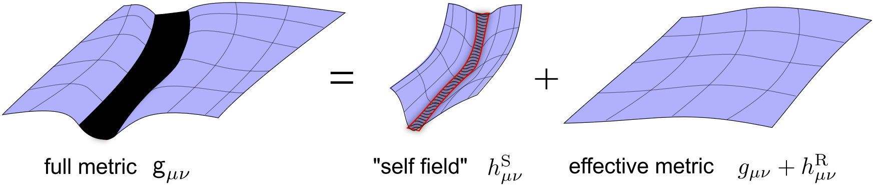

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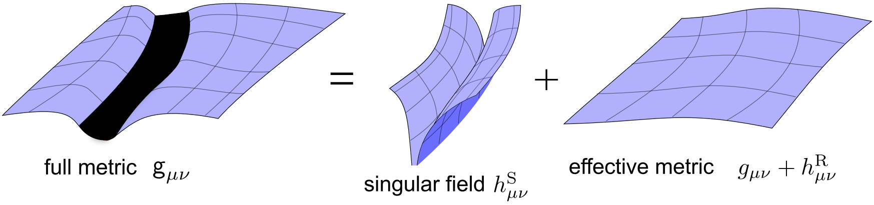

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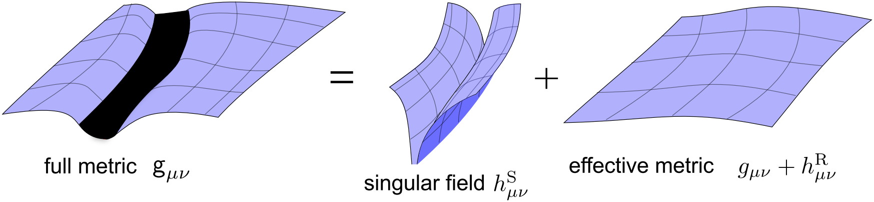

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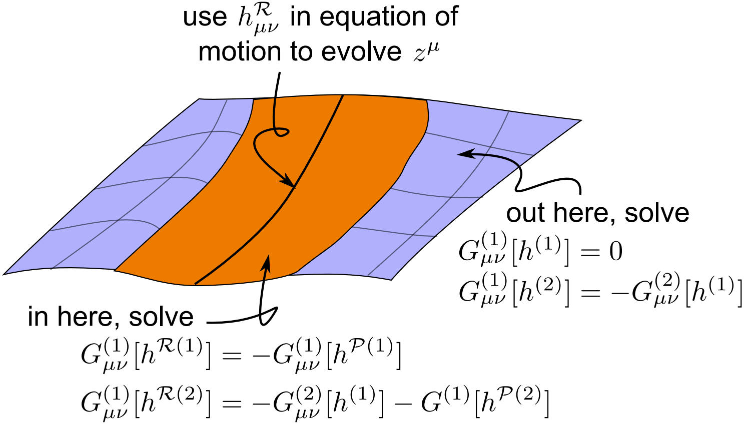
Replacing an object with a puncture

- truncate local expansion of $h^{\mathrm{S}(n)}_{\mu\nu}$, call it the puncture $h^{\mathcal{P}(n)}_{\mu\nu}$
- solve field equations for *residual field*

$$h^{\mathcal{R}(n)}_{\mu\nu} := h^{(n)}_{\mu\nu} - h^{\mathcal{P}(n)}_{\mu\nu}$$

- move the puncture with eqn of motion (using $\partial h^{\mathcal{R}(n)}_{\mu\nu}|_\gamma = \partial h^{\mathrm{R}(n)}_{\mu\nu}|_\gamma$)



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Point particle approximation

The following problems are equivalent:

- A FBVP:

$$G^{(1)}_{\mu\nu}[h^{(1)}] = 0 \quad \text{for } x^\mu \neq z^\mu$$
$$h^{(1)}_{\mu\nu} = h^{\mathrm{S}(1)}_{\mu\nu} + h^{\mathrm{R}(1)}_{\mu\nu} \quad \text{for } x^\mu \text{ near } z^\mu$$

- A puncture scheme:

$$G^{(1)}_{\mu\nu}[h^{\mathcal{R}(1)}] = -G^{(1)}_{\mu\nu}[h^{\mathcal{P}(1)}] := S^{\mathrm{eff}}_{\mu\nu} \quad \text{for all } x^\mu$$

- A point particle equation:

$$G^{(1)}_{\mu\nu}[h^{(1)}] = 8\pi \int u_\mu u_\nu \frac{\delta^4(x^\alpha - z^\alpha)}{\sqrt{-g}} d\tau := 8\pi T^{(1)}_{\mu\nu}$$

(coupled to EOM for z^μ in each case).

An aside

These are also equivalent:

$$G^{(1)}_{\mu\nu}[h^{\mathcal{R}(1)}] = -G^{(1)}_{\mu\nu}[h^{\mathcal{P}(1)}] := S^{\rm eff}_{\mu\nu}$$

$$G^{(1)}_{\mu\nu}[h^{\mathcal{R}(1)}] = 8\pi T^{(1)}_{\mu\nu} - G^{(1)}_{\mu\nu}[h^{\mathcal{P}(1)}] := S^{\rm eff}_{\mu\nu}$$

An aside

These are also equivalent:

ordinary derivatives

$$G^{(1)}_{\mu\nu}[h^{\mathcal{R}(1)}] = -G^{(1)}_{\mu\nu}[h^{\mathcal{P}(1)}] := S^{\rm eff}_{\mu\nu}$$

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distributional derivatives

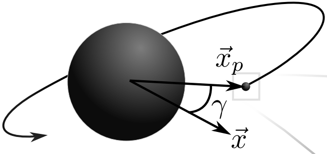
Recovering the regular field

- If we solve the point-particle equation for $h^{(1)}_{\mu\nu}$, we need to recover $h^{\mathrm{R}(1)}_{\mu\nu}$ from it
- We could use

$$h^{\mathrm{R}(1)}_{\mu\nu}(z) = \lim_{x\to z}[h^{(1)}_{\mu\nu}(x) - h^{\mathcal{P}(1)}_{\mu\nu}(x)]$$
$$\partial_\rho h^{\mathrm{R}(1)}_{\mu\nu}(z) = \lim_{x\to z}[\partial_\rho h^{(1)}_{\mu\nu}(x) - \partial_\rho h^{\mathcal{P}(1)}_{\mu\nu}(x)]$$

  etc. But hard to implement
- Instead, expand fields in spherical harmonics and subtract at level of indivdual ℓ modes

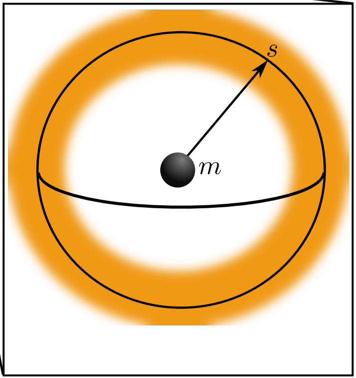
Mode-sum regularization [Barack & Ori and others]



- consider Coulomb potential:

$$\varphi = \frac{q}{|\vec{x} - \vec{x}_p|}$$
$$= \frac{q}{r_>} \sum_\ell \left(\frac{r_<}{r_>}\right)^\ell P_\ell(\cos\gamma)$$

- individual ℓ modes are finite at particle
  —divergence comes from sum over ℓ



Mode-sum regularization continued

$$
\begin{aligned}
h^{\mathrm{R}(1)}_{\mu\nu}(z) &= \lim_{x\to z}\left[h^{(1)}_{\mu\nu}(x) - h^{\mathrm{S}(1)}_{\mu\nu}(x)\right] \\
&= \lim_{x\to z}\sum_{\ell m}\left[h^{\ell m}_{\mu\nu}(t,r)Y_{\ell m}(\theta,\phi) - h^{\mathrm{S},\ell m}_{\mu\nu}(t,r)Y_{\ell m}(\theta,\phi)\right] \\
&= \lim_{r\to r_p}\sum_{\ell m}\left[h^{\ell m}_{\mu\nu}(t,r)Y_{\ell m}(\theta_p,\phi_p) - h^{\mathrm{S},\ell m}_{\mu\nu}(t,r)Y_{\ell m}(\theta_p,\phi_p)\right] \\
&= \lim_{r\to r_p}\sum_{\ell}\left[h^{\ell}_{\mu\nu}(t,r) - h^{\mathrm{S},\ell}_{\mu\nu}(t,r)\right] \\
&= \sum_{\ell}\left[h^{\ell}_{\mu\nu}(t,r_p) - h^{\mathrm{S},\ell}_{\mu\nu}(t,r_p)\right]
\end{aligned}
$$

Regularization parameters

- In Lorenz gauge, $h^{\mathrm{S},\ell}_{\mu\nu}(t,r_p) = B_{\mu\nu} + C_{\mu\nu}/L + O(1/L^2)$ at large $L = \ell + 1/2$
- So

$$\begin{aligned} h^{\mathrm{R}(1)}_{\mu\nu}(z) &= \sum_\ell \left[h^\ell_{\mu\nu}(t,r_p) - h^{\mathrm{S},\ell}_{\mu\nu}(t,r_p)\right] \\ &= \sum_\ell \left[h^\ell_{\mu\nu}(t,r_p) - B_{\mu\nu} - C_{\mu\nu}/L\right] \\ &\quad - \sum_\ell \left[h^{\mathrm{S},\ell}_{\mu\nu}(t,r_p) - B_{\mu\nu} - C_{\mu\nu}/L\right] \\ &:= \sum_\ell \left[h^\ell_{\mu\nu}(t,r_p) - B_{\mu\nu} - C_{\mu\nu}/L\right] - D_{\mu\nu} \end{aligned}$$

- Method works for any $\mathcal{Q}[h^{\mathrm{R}(1)}]$, where $\mathcal{Q}$ is linear differential operator

Main takeaways

- Singularities introduced in a controlled way, to replace a FBVP with a simpler, equivalent problem
- Regularization prescriptions recover specific finite quantities defined *prior* to the replacement
- Picture emerges of a test mass in an effective metric

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