Conformal and related techniques for applications in GR General Relativity

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background:

G-., Kopinski, Higher fundamental forms . . . asymptotically de Sitter spacetimes *CQG* 2022

Curry, **G-.** $\cdots$ Conformal Geometry $\cdots$ GR$\cdot\cdots$, LMS Series, Cambridge, arXiv:1412.7559

Čap, **G-.** Hammerl: Holonomy reductions etc, *Duke Math. J.* **163** (2014) 1035–1070.

G-. *J. Geom. Phys.*, **60** (2010), 182–204.

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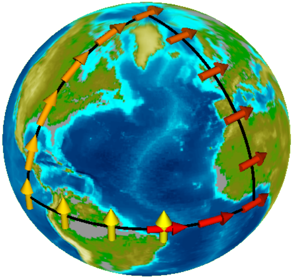
Warsaw GR Soc School, Poland 2023

Part 0 • Basic geometry – pseudo-Riemannian geometry and space-time geometry,

Part I: • The motivation for, and classical approach to, conformal compactification.

Part II: • Conformal geometry and tractor calculus.

• The geometry of scale and it's use to understand and extend the theory of space-time compactification.



Warning $\boxed{d=n+1}$

Pseudo-Euclidean space

An obvious variant of Euclidean space $\mathbb{E}^n$ arises by replacing the usual dot product on $\mathbb{R}^n$ with a **pseudo-Euclidean metric**,

$$\eta(x,y) = \sum_{i=q+1}^{n=p+q} x_i y_i - \sum_{i=1}^{q} x_i y_i, \qquad \text{signature } (p,q) \rightsquigarrow \mathbb{E}^{p,q}.$$

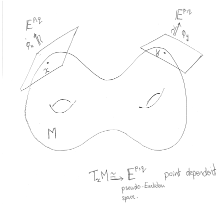
E.g. $\mathbb{M}^n = \mathbb{E}^{n-1,1}$ arises by replacing $(\star)$ with the **signature** $(\boldsymbol{n-1,1})$ **Minkowski** inner product:

$$\eta(x,y) = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Then $\exists$ **null vectors:** $x \neq 0$ s.t. $\eta(x,x) = 0$, in fact a **null cone** of such. This is fixed by the group $O(\eta) \cong O(n-1,1)$ preserving η. So $\mathbb{E}^{3,1}$ models a **space-time** geometry where the "speed of light is the same in all frames" as required by Michelson–Morley experiments. Hence Einstein's **special relativity**.

Riemannian and pseudo-Riemannian geometry

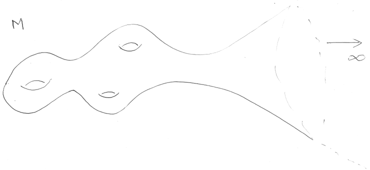
Above is vastly generalised by **pseudo-Riemannian geometry** = manifold M plus point dependent pseudo-Euclidean structure:



That is M is equipped with a **metric** g = point dependent inner product, of signature (p,q), on $TM = \cup_{x\in M}(T_x M)$ – tangent bundle. E.g. Idea of **GR** is that space-time is well-modelled by a Lorentzian signature pseudo-Riemannian manifold.

Local geom./analysis: g determines a unique **connection** ∇^g on TM that sat. $\nabla^g g = 0$ – i.e. way of transporting vectors along curves. Then $g, \rightsquigarrow \nabla^g \rightsquigarrow$ notion of **curvature**, **invariants** and natural operators/eqns - e.g. geometric Laplacian/d'Alembertian $\Delta^g := g^{ab}\nabla_a\nabla_b$; **Einstein eqn** $Ric = \lambda g$.

A general question: Suppose we have an infinite space-time - or space (i.e. non-compact manifold, geodesically complete):

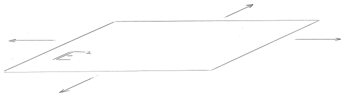


How do we deal with the "far region"? Can we make a notion of "infinity" that is mathematically useful? If so what geometry does it have? Are there many ways to do such things, or is any success essentialy unique?

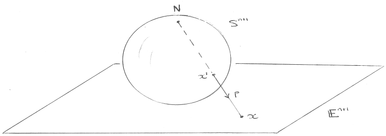
A **Compactification** of a non-compact ("large") topological space M is an embedding of M as a dense subset of a compact ("small") space $\overline{M}$: So $\boxed{M \hookrightarrow \overline{M} \text{ injective cts and a homeo. onto its image}}$.

Example: Euclidean space

Problem: Euclidean d-space $\mathbb{E}^d$ is a big place for certain problems. How can we effectively treat **all of it** mathematically?



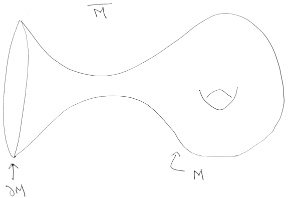
One solution: **strategically add points**! $\mathbb{E}^d$ is **non-compact**. Idea: add points (not too many?) somehow so the result is **compact**. Observe that the d-sphere S^d is $\mathbb{E}^d \cup \{\text{one point}\}$:



Stereographic projection $p : S^d \setminus \{\mathrm{N}\} \to \mathbb{E}^{\mathrm{d}}$ a diffeo. $S^d =$ **one point conformal compactification** of $\mathbb{E}^d$.

Compactification: $M \hookrightarrow \overline{M}$ smooth injective, M open dense. (In general $\overline{M}$ may be a manifold with boundary, a manifold with corners,) . . . **Question:** What is a right way to do this when geometry is involved?

In many simple cases the result is a **manifold with boundary** $\overline{M}$ so that M is the interior and ∂M has codimension 1.



Questions: How do we find the geometry on ∂M?
Boundary calculus: Relate the geometries/fields on ∂M and M?
Applications: 1. Discovery new links between the different geometries on ∂M and M; A more geometric and conceptual approach to the PDE boundary problems; scattering and non-local operators; . . .

Example: A compactification of Minkowski space

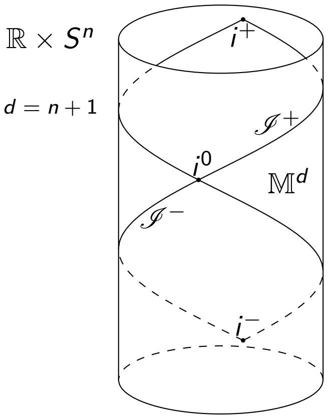


Figure: The standard embedding of d-dimensional Minkowski space $\mathbb{M}^d$ into the **Einstein cylinder**. This is **conformal**: $g_{\mathrm{Mink}} = \Omega^2 g_{Lorentzian\ cyln}$

Questions: Is this essentially the only way to conformally compactify $\mathbb{M}$? Is it forced that $i^{\pm}$ and i^0 are points? That $\mathscr{I}$ is an open subset of $\partial\mathbb{M}$? Why is $\partial\mathbb{M}$ different to $\partial\mathbb{E}^{n+1}$?

Penrose's "generalisation" and conformal infinity

Definition

A smooth (time- and space-orientable) spacetime (M_+, g_+) is called **asymptotically simple** if there exists another smooth Lorentzian manifold $(\overline{M}, \overline{g})$ such that

1. M_+ is an open submanifold of $\overline{M}$ with smooth boundary $\partial M_+ = \mathscr{I}$;
2. there exists a smooth scalar field Ω on M, such that $\overline{g} = \Omega^2 g_+$ on M_+, and so that $\Omega = 0$, $\mathrm{d}\Omega \neq 0$ on $\mathscr{I}$;
3. every null geodesic in $\overline{M}$ acquires a future and a past endpoint on $\mathscr{I}$.

An asymptotically simple spacetime is called **asymptotically flat** if in addition $\mathrm{Ric}^{g_+} = 0$ in a neighbourhood of $\mathscr{I}$.

Questions: How would we re-discover the Einstein cylinder compactification or this useful definition? Treat other geometries similarly?

Conformal compactification of $\mathbb{H}^{n+1}$ – the Poincaré ball

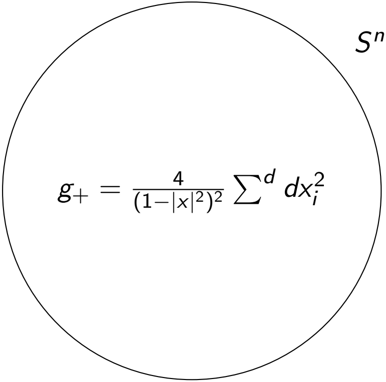
Escher's circle limit

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 + \partial\mathbb{H}^2$$

The embedding gives the compactification

$\mathbb{H}^d$ embedded conformally in Euclidean $\mathbb{E}^d$ – Poincaré-Ball

$S^n = \partial\mathbb{H}^{n+1}$

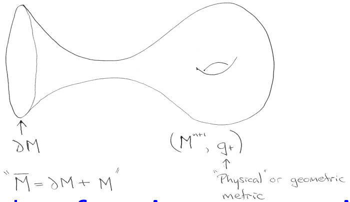


$$\overline{\mathbb{H}^d} = \mathbb{H}^d + \partial\mathbb{H}^d$$

Conformal compactification

Henceforth in these talks, a **conformal compactification** of a pseudo-Riemannian manifold (M^d, g_+) is a smooth manifold $\overline{M}$ with boundary ∂M s.t.:

- $\exists\ \overline{g}$ on $\overline{M}$, with
- $g_+ = r^{-2}\overline{g}$, where r a **defining function** for ∂M. (i.e. $\partial M = r^{-1}(0)$ and dr non-vanishing on ∂M.)



$\Rightarrow$ canonical **conformal structure on boundary**: $(\partial M, [\overline{g}|_{\partial M}])$ (where dr not null).

- g_+ then called **conformally compact**. I will say it is a **Poincaré-Einstein** metric if also g_+ is **Einstein** in the sense $\mathrm{Ric}^d = \lambda g$.

Conformal geometry

Question: Why is $\partial\mathbb{E}^{n+1}$ =one point, whereas $\partial\mathbb{M}^{n+1}$ ="a cone"? What geometry does latter have? How do we generalise to other infinite (= complete non-compact) manifolds? We need:

Conformal geometry = geometry with "angle but not length".

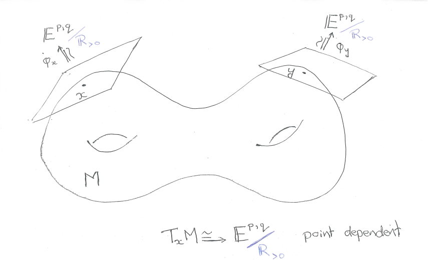
More precisely: a **conformal manifold** consists of

$$(M, \boldsymbol{c})$$

where M is a smooth manifold (dim $d = n+1 \geq 3$) and $\boldsymbol{c}$ is an **equivalence class** of metrics

$$g \sim g' \quad \text{if} \quad g' = f^2 g$$

where f a positive function.



Not so crazy – Conformal geometry in math and phys

Physics: The equations of electromagnetism – the Maxwell equations $\delta^* F = 0$ – are not just Lorentz invariant but conformally invariant (Bateman 1909).

• More generally the Yang-Mills equations that govern weak and strong force – are also conformally invariant in dimension 4.

• AdS/CFT correspondence of String Theory.

Mathematics: Complex analysis – Riemann surfaces are 2-d conformal manifolds.

• The geometry of smooth domain boundaries in $\mathbb{C}^n$ – CR geometry (– "almost conformal").

• Most important: g determines $[g] = \boldsymbol{c}$ – has a deep role in aspects pseudo-Riemannian geometry. E.g. Yamabe problem and its generalisations (uniformisation), Symmetry and dynamics . .

• Scattering.

• A tool for geometric compactification.

Part II: Conformal geometry and the geometry of scale

Because there is no distinguished metric on $(M^d, \mathbf{c})$ an important role is played by the **density bundles**. Note $(\Lambda^d TM)^2$ is an oriented real line bundle $\mathcal{K}$. We write $\mathcal{E}[w]$ for the roots

$$\mathcal{E}[w] = \mathcal{K}^{\frac{w}{2d}}, \qquad \text{so} \qquad \mathcal{K} = \mathcal{E}[2d],$$

$\mathcal{E}[0] := \mathcal{E}$ (the trivial bundle with fibre $\mathbb{R}$), and $\mathcal{E}_+[w]$ for the positive elements. With this notation there is tautologically a **conformal metric**

$$\boldsymbol{g} \in S^2 T^* M[2], \qquad \text{so that} \qquad g^\sigma := \sigma^{-2}\boldsymbol{g} \in \mathbf{c}, \quad \sigma \in \Gamma(\mathcal{E}_+[1]),$$

and $\otimes^d \boldsymbol{g} : \left(\Lambda^d TM\right)^2 \xrightarrow{\simeq} \mathcal{K} = \mathcal{E}[2d]. \quad (\star)$

There is 1-1 relation between sections σ of $\mathcal{E}_+[1]$ and metrics g^σ in $\mathbf{c}$ (via $g^\sigma := \sigma^{-2}\boldsymbol{g} \in \mathbf{c}$), and we call

$$\sigma \in \Gamma(\mathcal{E}_+[1]) \qquad \text{a } \textbf{strict scale}.$$

The LC connection ∇^g, for $g \in \mathbf{c}$, acts on $\mathcal{K}$ through $(\star)$,

$$\text{whence} \quad \nabla^g \boldsymbol{g} = 0 \ \rightsquigarrow \ \nabla^{g^\sigma}\sigma = 0.$$

So σ **is parallel for the Levi-Civita connection it determines.**

Problem: Conformal geometry is not as rigid as pseudo-Riemannian geometry. This has benefits but is also a big problem. How do we "get a handle on it"? The geometry, the geometric analysis, etc :

There is **no metric** on TM and **no connection** on TM.

On $(M, \boldsymbol{c})$, for each $g \in \boldsymbol{c}$ there is ∇^g. But if $\widehat{g} = f^2 g$ then for $\xi, \eta \in \mathfrak{X}(M)$

$$\nabla^{\widehat{g}}_\xi \eta = \nabla^g_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi - g(\xi,\eta)\Upsilon^\sharp \quad \text{where} \quad \Upsilon = \nabla \log f .$$

$$\text{so e.g.} \quad \Delta^{\hat{g}} u = f^{-2}\left(\Delta^g u + (n-2)\Upsilon^c \nabla_c u\right).$$

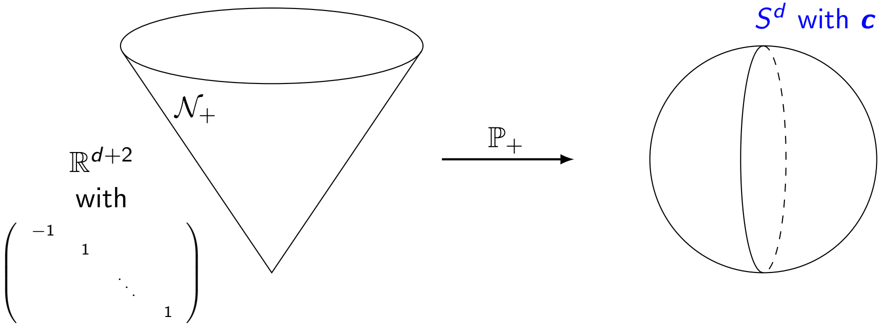
"Error terms" as on the RHS here increase exponentially with order of calculations.

Solution: It turns out that:

There is **a metric** and **a connection** on $\mathcal{T} = TM \oplus$(a bit).

The conformal sphere = the (Riemannian) model

Conformal sphere is the ray projectivisation of forward null cone:



- Affine parallel transport on $\mathbb{R}^{d+2}$ gives a **conformally invariant connection** on $(S^d, \mathbf{c})$ (!!)

This **tractor connection** $\nabla^{\mathcal{T}}$ is on a v. bundle $\mathcal{T}$ where, at each $x \in S^d = M$, $\mathcal{T}_x \cong T_p\mathbb{R}^{d+2} \cong \mathbb{R} \oplus T_xM \oplus \mathbb{R}$, for $p \in x \subset \mathcal{N}_+$.
$\mathcal{T}$ has a Lorentzian **metric** h that is preserved by $\nabla^{\mathcal{T}}$: $\nabla^{\mathcal{T}} h = 0$.

The conformal $S^1 \times S^n$ = the Lorentzian model

Now take $\mathbb{R}^{d+2}$ with bilinear form

$$h = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

The null cone/quadric is

$$x_0^2 + x_1^2 = x_2^2 + \cdots x_{d+1}^2.$$

The ray projectivisation has a **Lorentzian sig. conformal structure** that includes the metric induced on the **section** $x_0^2 + x_1^2 = 1 = x_2^2 + \cdots x_{d+1}^2$ (so $S^1 \times S^n$) with its induced Lorentzian signature metric.

So the tractor bundle $(\mathcal{T}, h, \nabla^{\mathcal{T}})$ on $(S^1 \times S^n, \boldsymbol{c})$ has signature $(d, 2)$. $\boxed{d = n+1}$

General curved conformal manifolds

On a general conformal manifold there is a canonical structure which generalises the above (Thomas 1926-31 and cf Cartan 1923, BEG 1989 (– cf also Penrose's local twistor transport for $d = 4$)):

Theorem

On a conformal manifold $(M, \mathbf{c})$ *of dimension* $d \geq 3$ *and signature* (p, q) *there is, canonically, a* **tractor bundle**

$$\mathcal{T} = \mathcal{E}[1] \oplus TM[-1] \oplus \mathcal{E}[-1]$$

with a **connection** *(i.e. parallel transport)* $\nabla^{\mathcal{T}}$, *and a* **signature** $(p+1, q+1)$ **metric** *h that is preserved by* $\nabla^{\mathcal{T}}$:

$$\nabla^{\mathcal{T}} h = 0.$$

There is also a canonical (or position) tractor $X \in \Gamma(\mathcal{T}[1])$ that gives the filtration of $\mathcal{T}$:

$$X : \mathcal{E}[-1] \to \mathcal{T} \qquad \text{and} \qquad X^* : \mathcal{T} \to \mathcal{E}[1].$$

The tractor connection

So although on a conformal manifold $(\overline{M}, \mathbf{c})$ there is no distinguished connection on TM – we have the conformally invariant **tractor bundle $\mathcal{T}$** and **connection $\nabla^{\mathcal{T}}$**. Given $\overline{g} \in \mathbf{c}$ this is given by

$$\mathcal{T} \stackrel{\overline{g}}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \qquad \mathcal{E}[1] := (\Lambda^d TM)^{\frac{2}{2d}}$$

$$\nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho) = (\nabla_a\sigma - \mu_a,\ \nabla\mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a\rho - P_{ab}\mu^b),$$

and $\nabla^{\mathcal{T}}$ preserves a conformally invariant **tractor metric** h

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b\mu^b = h(V, V).$$

There is also a second order **Thomas operator**:

$$\Gamma(\mathcal{E}[w]) \in f \mapsto D_A f \stackrel{\overline{g}}{=} \begin{pmatrix} (d+2w-2)wf \\ (d+2w-2)\nabla_a f \\ -(\Delta f + wJf) \end{pmatrix}$$

where J is $\mathrm{trace}^{\overline{g}}(P_{ab})$, so a number times $\mathrm{Sc}(\overline{g})$.

Parallel standard tractors

Note that from the formula

$$\nabla_a^{\mathcal{T}}(\sigma, \mu_b, \rho) = (\nabla_a\sigma - \mu_a,\ \nabla\mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a\rho - P_{ab}\mu^b),$$

if $I_A \stackrel{g}{=} (\sigma, \mu_a, \rho)$ is a parallel tractor then $\mu_a = \nabla_a\sigma$, and $\rho = -(\Delta\sigma + w\mathrm{J}\sigma)$. This gives the first statement of:

Proposition

I parallel implies $I_A = \frac{1}{d}D_A\sigma$. So $I \neq 0 \Rightarrow \sigma$ is nonvanishing on an open dense set $M_{\sigma\neq 0}$. On $M_{\sigma\neq 0}$, $g^o = \sigma^{-2}\mathbf{g}$ is Einstein. Conversely if $g^o = \sigma^{-2}\mathbf{g}$ is Einstein then $I := \frac{1}{d}D\sigma$ is parallel.

Proof.

On $M_{\sigma\neq 0}$ we have locally $\pm\sigma \in \Gamma(\mathcal{E}_+[1])$ so $\mu_a = \nabla_a\sigma = 0$ for $\nabla = \nabla^{g^\sigma}$. Thus

$$P_{ab} + \frac{\rho}{\sigma}\boldsymbol{g}_{ab} = 0.$$

The converse is easy. □

So we say $(M, \mathbf{c})$ with parallel scale tractor $I := \frac{1}{d}D\sigma \neq 0$ is **almost Einstein**. $\Leftrightarrow \nabla_{(a}\nabla_{b)}\sigma + P_{(ab)_0}\sigma = 0, \quad \sigma \neq 0.$

A canonical stratification – strata called "Curved orbits"

Concerning $M_0 = \mathcal{Z}(\sigma)$. (Here and throughout $I^2 = I^A I_A$.)

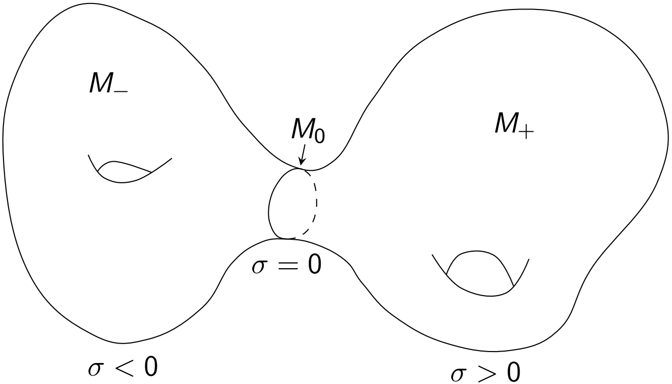
Theorem: An almost Einstein manifold $(M, \mathbf{c}, I)$ (of any signature) is stratified according to the strict sign of $\sigma = I_A X^A$: $M = M_+ \cup M_0 \cup M_-$ and $(I^2 = -\frac{Sc^{g_\sigma}}{d(d-1)}$ and so)

- If $I^2 \neq 0$ (i.e. g^o Einstein and not Ricci flat) then $M_0 := \mathcal{Z}(\sigma)$ is either empty or is a smoothly embedded separating hypersurface.
- If $I^2 = 0$ (i.e. g^o Ricci flat) then $M_0 := \mathcal{Z}(\sigma)$ is either empty or, **after excluding isolated points** from $\mathcal{Z}(\sigma)$, is a **smooth embedded separating hypersurface**.
- $M_+ \cup M_0$ is a conformal compactification of (M_+, g^o). Similar $M_- \cup M_0$. As $g^o = \sigma^{-2}\boldsymbol{g}$ and $\nabla\sigma$ is nowhere zero on the hypersurface (parts of) M_0

Proof: Last • as $g^o = \sigma^{-2}\boldsymbol{g}$ and $\nabla\sigma$ is nowhere zero on the hypersurface (parts of) M_0. The rest follows by easy local analysis. Also alternatively general curved orbit theorem in L3.

The picture so far

Thus if e.g. $I^2 \neq 0$ and I_A is **parallel** we have the picture:



And the $M \setminus M_\pm$ are **conformally compact** and hence **Poincaré-Einstein (PE)**. Conversely all Poincaré-Einstein manifolds arise this way. (The almost Einstein manifold M is a glueing of the PE parts along their conformal infinities.)

Models!? Pause for thought

If we believe all this then we should be able to understand/recover the models we discussed.

The one point compactification of $\mathbb{E}^d$!?

The Einstein cylinder!?

We will see later that these models actually tell us a lot about the structure and asymptotics even the general/curved setting.

First the most famous model **again**:

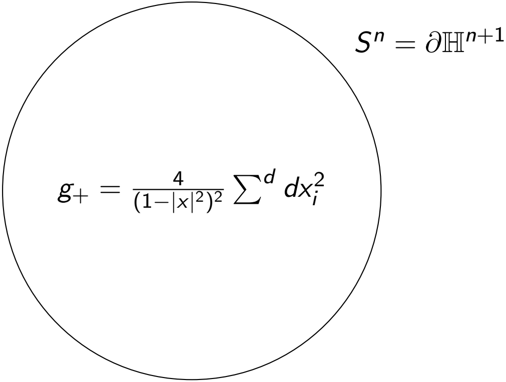
Conformal compactification of $\mathbb{H}^{n+1}$ – the Poincaré ball

Escher's circle limit

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 + \partial\mathbb{H}^2$$

The embedding gives the compactification

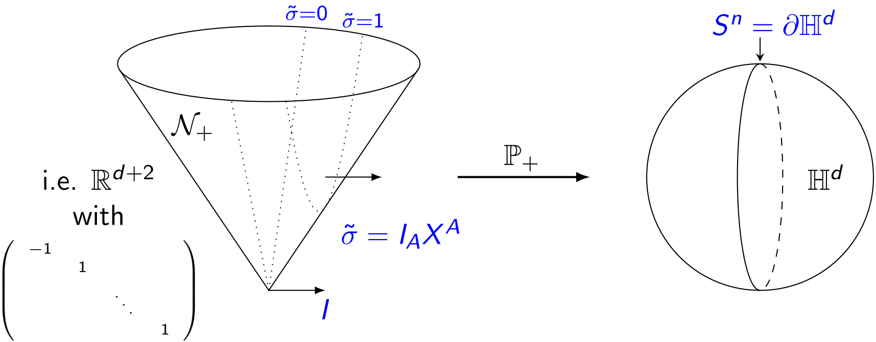
$\mathbb{H}^d$ embedded conformally in Euclidean $\mathbb{E}^d$ – Poincaré-Ball



$$\overline{\mathbb{H}^d} = \mathbb{H}^d + \partial\mathbb{H}^d$$

Poincaré compactification via $\mathbb{P}_+(\mathit{nullcone})$

Conformal compactification of $\mathbb{H}^d$ by **symmetry breaking**:



$S^d = \mathbb{P}_+(\mathcal{N}_+ \subset \mathbb{R}^{d+2} \setminus \{0\})$ is model of flat conformal geometry.

$G := SO_o(d+1,1)$ acts transitively. $I \in \mathbb{R}^{d+2}$, spacelike $h(I,I) = 1$

I constant is a parallel tractor. $\therefore$ Right hemi. where $X^A I_A = \tilde\sigma \geq 0$ is conf. compactification $\overline{M}_c$ of $\mathbb{H}^d$; $\sigma = 0$ conformal ∞ with conformal str.

Stereographic proj. of this $\mathbb{H}^d$ recovers Poincaré Ball.

Minkowski, de Sitter, and AdS

Recall that the Lorentzian conformally flat model is $(S^1 \times S^n, \boldsymbol{c})$ arising as the ray projectivisation

$$S^1 \times S^n = \mathbb{P}_+(\{x_0^2 + x_1^2 = x_2^2 + \cdots x_{d+2}^2\}) \quad \leftarrow G = S0_o(d,2) \text{acts trans}^{\text{y}}$$

of the null quadric in

$$\mathbb{R}^{d+2} \quad \text{with bilinear form} \quad h = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \boxed{d := n+1}$$

Then $I_A \in \mathbb{R}^{d+2}$ – constant $\Leftrightarrow$ **parallel tractor** with:

• $I^2 = -1$: Two copies of **conf. compactified de Sitter** on *interval* $\times S^n$ – as $(\nabla\sigma)^2 = -1$ on zero locus $\rightsquigarrow \mathcal{Z}(\sigma)$ spacelike;

• $I^2 = 0$: Two copies of **conf. compactified Minkowski**, as in Ein. cyl.. Singular boundary points where X (anti-)parallel to I

• $I^2 = 1$: Two copies of **c.c. anti-de Sitter** on $S^1 \times$ *hemisphere*.

– Respectively $H =: S0_o(d,1)$, Poincaré grp, and $S0_o(d-1,2)$ orbit decompositions.

$d = 4$ Via coordinates (Jarek Kopiński calcs)

$\mathbb{R}^6$ equipped with

$$h = -du^2 - dv^2 + dx^2 + dy^2 + dz^2 + dw^2 \Leftrightarrow h = -dR^2 - R^2 dt^2 + dr^2 + r^2 g_{S^3}$$

where $u = R \sin t$, $v = R \cos t$ and g_{S^3} is the S^3 metric written with the use of angles (ψ, θ, φ). Then the null cone on $r = R$ hypersurface with the induced degenerate metric h_{ab} given by

$$h = R^2 \left(-dt^2 + g_{S^3} \right). \tag{1}$$

The conformal metric g_{ab} on $S^1 \times S^3$ (the $\mathbb{R}_+$-ray projectivisation of the null cone) arises as a restriction of h_{AB} to the tangent vectors to the cone which are lifts to this cone of the vectors on $S^1 \times S^3$. Each choice of the constant vector field I on M induces an Einstein scale $\sigma = I^A X_A$, Let I be proportional to the timelike coordinate vector ∂_v, i.e.

$$I_{dS} := \sqrt{\frac{\Lambda}{3}}\partial_v, \quad \Lambda > 0. \tag{2}$$

We have $I_{dS}^2 = -\frac{\Lambda}{3}$ $(R_{dS} = 4\Lambda)$. The induced Einstein scale is

$$\sigma_{dS} := \sqrt{\frac{\Lambda}{3}}v = \sqrt{\frac{\Lambda}{3}}R\cos t \tag{3}$$

and splits $S^1 \times S^3$ into two regions with the common boundary being the zero locus of σ_{dS}, i.e. the two copies of the compactified de Sitter spaces glued together along their spacelike infinities. The metric g_{dS} of each copy has the form

$$g_{dS} = \sigma_{dS}^{-2}\mathbf{g} = \frac{3}{\Lambda\cos^2 t}\left(-dt^2 + g_{S^3}\right) \tag{4}$$

spacelike I – anti-de Sitter case

Let

$$I_{AdS} := \sqrt{\frac{|\Lambda|}{3}}\partial_w, \quad \Lambda < 0. \tag{5}$$

We have $I_{AdS}^2 = \frac{|\Lambda|}{3}$, i.e. $R_{AdS} = 4\Lambda$. The induced Einstein scale

$$\sigma_{AdS} := \sqrt{\frac{|\Lambda|}{3}} R \cos\psi \tag{6}$$

splits $S^1 \times S^3$ into two copies of the compactified anti-de Sitter space glued along timelike infinities. The Einstein metric is given by

$$g_{AdS} = \frac{3}{|\Lambda| \cos^2\psi}\left(-dt^2 + g_{S^3}\right) \tag{7}$$

null I – Minkowski case

Let

$$I_{Mi} := \partial_v + \partial_w, \tag{8}$$

i.e. $I_{Mi}^2 = 0$. This choice defines a split of $S^1 \times S^3$ into two copies of compactified Minkowski space glued along their null infinities.

The Ricci flat metric is

$$\begin{aligned} g_{Mi} &= \frac{1}{(\cos t + \cos\psi)^2}\left(-dt^2 + g_{S^3}\right) \\ &= \frac{1}{4\cos^2\left(\frac{t+\psi}{2}\right)\cos^2\left(\frac{t-\psi}{2}\right)}\left(-dt^2 + g_{S^3}\right) \end{aligned} \tag{9}$$

The END of lecture one

Lecture Two

Refs:

G-.; Nurowski, Obstructions to conformally Einstein metrics in n dimensions. J. Geom. Phys. (2006)

G-. Almost Einstein and Poincaré-Einstein manifolds in Riemannian signature. J. Geom. Phys. (2010)

G-. ; Waldron, Boundary calculus for conformally compact manifolds. Indiana Univ. Math. J. (2014)

Curry, G-. An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity, 86–170, in Asymptotic Analysis in General Relativity, LMS Series, Cambridge, 2018. arXiv:1412.7559

Plan of Lecture Two

Part I. The geometry of scale.

We return to and further develop the picture that was emerging from L1.

Part II. Hypersurfaces in conformal geometry – first taste
– with a view to treating spacetime conformal infinities (and other boundaries) as well as their asymptotics.

Part III. Applications of the conformal approach to understanding the space-time boundary (at infinity) geometry – first taste.

Part IV. Applications of the conformal approach to boundary problems and scattering.

Part I. The geometry of scale

Let us return now to the picture that was emerging from L1. We learned there that we could understand Poincare-Einstein manfolds and similar as conformal manfiolds plus scale $\sigma \in \Gamma(\mathcal{E}[1])$ given by

$$\sigma := X^A I_A, \quad \text{where} \quad I_A \quad \text{is parallel} \Rightarrow I_A = \frac{1}{d} D_A \sigma.$$

Now we want to understand more general conformally compact manifolds in similar way. Somehow!

On conformal $(M, \mathbf{c})$ we will use the scale tractor

$$I_A := \frac{1}{d} D_A \sigma \qquad \sigma \in \Gamma(\mathcal{E}[1])$$

now **not** assuming I is parallel.

Almost pseudo-Riemannian geometry

For convenience we say that a structure

$$(M^d, \mathbf{c}, \sigma) \quad \text{where} \quad \sigma \in \Gamma(\mathcal{E}[1])$$

is **almost pseudo-Riemannian** if the tractor

$$I_A := \frac{1}{d} D_A \sigma \qquad \textbf{is nowhere zero} \overset{def.}{\leftrightarrow} I \text{ is a } \textbf{scale tractor}$$

Note then that σ is non-zero on an open dense set, since $D_A\sigma$ encodes part of the 2-jet of σ. So on an almost pseudo-Riemannian manifold there is the pseudo-Riemannian metric $g^o = \sigma^{-2}\mathbf{g}$ on the same open dense set. We will sometimes mention I instead of $\sigma = X^A I_A$ and refer to $(M, \mathbf{c}, I)$ as an almost pseudo-Riemannian manifold. Below we will see:

Lemma

A conf. compact mfld is an almost Riemannian manifold $(\overline{M}, \mathbf{c}, \sigma)$ *with boundary* $(\overline{M} = M_+ \cup \partial M_+)$ *such that* σ **defines** ∂M_+

σ **defines** ∂M_+ means $\sigma = \tau r$, $\tau \in \Gamma(\mathcal{E}_+[1])$, r defining fn for ∂M.

Now from $I_A = (\sigma,\ \nabla_a\sigma, -\frac{1}{d}(\Delta\sigma + J\sigma))$ and the metric we have

$$h^{AB} I_A I_B = I^A I_A =: I^2 \stackrel{g}{=} \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) - \frac{2}{d}\sigma(\mathrm{J} + \Delta)\sigma \qquad (10)$$

where g is any metric from $\mathbf{c}$ and ∇ its Levi-Civita connection. This is well-defined everywhere on an almost pseudo-Riemannian manifold. Where σ is non-zero, it computes

$$I^2 = -\frac{2}{d}\sigma^2\mathrm{J} = -\frac{2}{d}\mathrm{J}^{g^o} = -\frac{\mathrm{Sc}^{g^o}}{d(d-1)} \quad \text{where} \quad g^o = \sigma^{-2}\boldsymbol{g}.$$

Thus I^2 gives a **generalisation of the scalar curvature** (up to a constant factor $-1/d(d-1)$); it is canonical and smoothly extends the scalar curvature to include the zero set of σ. We shall use the term **ASC manifold** (where ASC means **almost scalar constant**) to mean an almost pseudo-Riemannian manifold with $I^2 = constant$. Since the tractor connection preserves h, then I parallel implies $I^2 = constant$. So an almost Einstein manifold is ASC, just as Einstein manifolds have constant scalar curvature.

Non-zero generalised scalar curvature.

Much of the almost Einstein curved orbit picture remains in the almost pseudo-Riemannian setting when I^2 is non-vanishing:

Theorem

Let $(M, \mathbf{c}, I)$ *be an almost pseudo-Riemannian manifold with* I^2 **nowhere zero**. *Then* $\mathcal{Z}(\sigma)$*, if not empty, is a smooth embedded separating hypersurface. This has a spacelike (resp. timelike) normal if* g^o *has negative scalar (resp. positive) scalar curvature. If* $\mathbf{c}$ *has Riemannian signature and* $I^2 < 0$ *then* $\mathcal{Z}(\sigma)$ *is empty.*

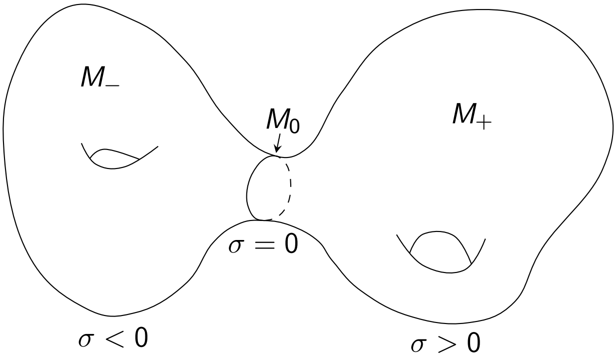
Key aspect of Proof.

From $I^2 \stackrel{g}{=} \mathbf{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) - \frac{2}{d}\sigma(\mathrm{J} + \Delta)\sigma$: Along $\mathcal{Z}(\sigma)$ we have

$$I^2 = \mathbf{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma).$$

in particular $\nabla\sigma$ is nowhere zero on $\mathcal{Z}(\sigma)$, and so σ is a **defining density**. Thus $\mathcal{Z}(\sigma)$ is a smoothly embedded hypersurface by the constant rank theorem.

The updated picture if $I_A = \frac{1}{n} D_A \sigma$ s.t $I^2 \neq 0$:



$(M, \boldsymbol{c})$ equipped with a **scale tractor** $I = \frac{1}{d} D\sigma$, with I^2 nowhere zero has I nowhere zero and so is almost pseudo-Riemanian. Where $\sigma = X^A I_A$ is nonzero (almost everywhere) there is the pseudo-Riemannian metric $g^o = \sigma^{-2}\boldsymbol{g}$, and σ is a defining density for the separating hypersurface $M_0 = Z(\sigma)$. Hence $M \setminus M_\pm$ is **conformally compact** with conf. infinity $(M_0, \boldsymbol{c}|_{M_0})$. Conversely all conformally compact manifolds arise this way*.

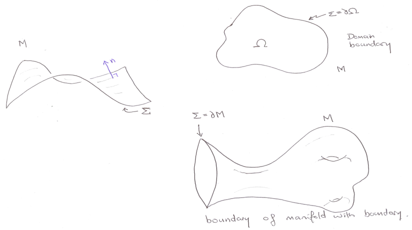
Summary and a look ahead

Moral: *Replace* (M, g) *with* $(M, \boldsymbol{c}, I)$ *where* I *is the* **scale tractor**. *This* **generalises our notion of geometry** *in a way that builds in the compactification data.*

E.g.$(*)$ $(\overline{M}, g_o)$ *a* **conformal compactification**, *with the scalar curvature bounded away from zero, means just* $(\overline{M}, \boldsymbol{c}, I)$ *where* $\overline{M} = M + \partial M$, $\partial M = \mathcal{Z}(\sigma)$ *and* I^2 *non-vanishing. (On* M, $g_o = \sigma^{-2}\boldsymbol{g}$*.)*

In pseudo-Riemannian geometry the metric g *produces geometric operators "*$\Delta^g = g^{ab} + \nabla_a + \nabla_b$*". Now we want to instead couple* σ *and* I_A *to conformal operators . . .*

Since ∂M *for us is codimension 1, we next we digress to look at hypersurface geometry in general.*



To treat asymptotics and boundary calculus we need to understand the mathematics of hypersurfaces.

Definition: **hypersurface** Σ in a manifold M means a smoothly embedded codimension 1 submanifold of $(M, \mathbf{c})$.

- we restrict to Σ with the property that the conormal field along Σ is nowhere null (i.e. to nondegenerate hypersurfaces). Then:
- restriction of any $g \in \mathbf{c}$ gives metric $\bar{g}$ on $\Sigma \rightsquigarrow \mathbf{c}$ induces $\bar{\mathbf{c}}$ on Σ.
- It is natural to work with a weight 1 co-normal n_a along Σ satisfying $\boldsymbol{g}^{ab} n_a n_b = \pm 1$.

Basic hypersurface invariants

For $g \in \mathbf{c}$, the **second fundamental form** K_{ab} is the restriction of $\nabla_a n_b$ to $T\Sigma \times T\Sigma \subset (TM \times TM)|_\Sigma$, where $\nabla = \nabla^g$; i.e.

$$K_{ab} := \nabla_a n_b \mp n_a n^c \nabla_c n_b \quad \text{along} \quad \Sigma.$$

This is not conformally invariant. But under a conformal rescaling, $g \mapsto \widehat{g} = e^{2\omega} g$, K_{ab} transforms according to

$$K^{\widehat{g}}_{ab} = K^g_{ab} + \overline{\mathbf{g}}_{ab} \Upsilon_c n^c, \qquad \text{where} \qquad \Upsilon = d\omega$$

Thus:

Proposition

The trace-free part of th second fundamental form

$$\mathring{K}_{ab} = K_{ab} - H \overline{\mathbf{g}}_{ab}, \quad \textit{where}, \quad H := \frac{1}{d-1} \overline{\mathbf{g}}^{cd} L_{cd}$$

is conformally invariant.

Here $d = n + 1$ is the dimension of the ambient manifold M.

The normal tractor

Evidently, under a conformal rescaling $g \mapsto \widehat{g} = e^{2\omega} g$, the **mean curvature** H^g transforms to $H^{\widehat{g}} = H^g + n^a \Upsilon_a$. Thus we obtain a conformally invariant section N of $\mathcal{T}|_\Sigma$

$$N_A \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ -H^g \end{pmatrix},$$

and $h(N, N) = \pm 1$ along Σ. This is the **normal tractor** of Bailey-Eastwood-G. Differentiating N tangentially along Σ using $\nabla^{\mathcal{T}}$, we obtain the following result.

Proposition (Conformal Shape operator)

$$\mathbb{K}_{aB} := \underline{\nabla}_a N_B \stackrel{g_{cb}}{=} \begin{pmatrix} 0 \\ \mathring{K}_{ab} \\ -\frac{1}{d-2} \nabla^b \mathring{K}_{ab} \end{pmatrix}$$

where $\underline{\nabla}$ is the pullback to Σ of the ambient tractor connection. Thus Σ is **totatally umbilic** *iff N is parallel along Σ.*

Conformal hypersurface calculus

The classical **Gauss formula**

$$\underline{\nabla}_a v^b = \overline{\nabla}_a v^b \mp n^b K_{ac} v^c \qquad v \in \Gamma(T\Sigma) \subset \Gamma(TM),$$

is the basis of pseudo-Riemannian hypersurface calculus.

We want the conformal analogue. First we need this:

Proposition (Branson-G., Grant)

There is a natural conformally invariant (isometric) isomorphism

$$\mathcal{T}|_\Sigma \supset N^\perp \xrightarrow{\simeq} \overline{\mathcal{T}} = \textit{std tractor bdle of } (\Sigma, \bar{\mathbf{c}}).$$

Proof.

Calculating in a scale g on M the tractor bundle $\mathcal{T}$, and hence also $N^\perp$, decomposes into a triple. Then the mapping of the isomorphism is

$$[N^\perp]_g \ni \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_b \mp H n_b \sigma \\ \rho \pm \frac{1}{2} H^2 \sigma \end{pmatrix} \in [\overline{\mathcal{T}}]_{\bar{g}}.$$

The tractor Gauss equation

The above reveals two connections on $\overline{\mathcal{T}} \cong N^{\perp}$ that we can compare. Namely the **intrinsic tractor connection** $\overline{\nabla}^{\overline{\mathcal{T}}}$ determined by $(\Sigma, \bar{\boldsymbol{c}})$, and the **projected ambient tractor connection** $\tilde{\nabla}$. The latter is defined by

$$\tilde{\nabla}_a U^B := \Pi^B_C(\Pi^c_a \nabla_c U^C) \qquad U \in \Gamma(N^{\perp}) \text{ extended arb. off } \Sigma$$

where Π^B_C and Π^c_a are the orthog. projections due to N and n. Including the tractor derivative of Π^B_C gives:

Proposition (Tractor Gauss formula – Stafford, Vyatkin)

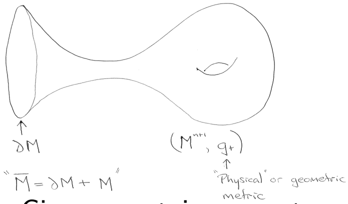
$$\underline{\nabla}_a V^B = \overline{\nabla}_a V^B \mp S_a{}^B{}_C V^C \mp \mathrm{N}^B \mathbb{K}_{aC} V^C,$$

where $\mathrm{S}_{aBC} = \overline{\mathbb{X}}_{BC}{}^c \mathcal{F}_{ac}$, *($\overline{\mathbb{X}}_{BC}{}^c$ an invariant bundle injector), and*

$$\mathcal{F}_{ab} = \frac{1}{d-2}\Big(W_{acbd} n^c n^d + \mathring{K}^2_{ab} - \frac{|\mathring{K}|^2}{2(d-1)} \bar{\boldsymbol{g}}_{ab}\Big).$$

Recall $\mathbb{K}_{aC} = \underline{\nabla}_a N_C$. This shows that $\mathcal{F}_{ab}$ is a conformal invariant of hypersurfaces. It is the so-called **Fialkow tensor**.

Given a conformally compact manifold:



Questions: Given a certain geometry of g_+ – e.g. Asymptotically de Sitter, asymptotically hyperbolic/de Sitter, Poincaré-Einstein, what can we say about the:

1. **Intrinsic geometry** of $(\partial M, \boldsymbol{c}|_{\partial M})$?
2. **Extrinsic geometry** of $(\partial M, \boldsymbol{c}|_{\partial M})$?
3. **Conformal geometry** of $(M, \boldsymbol{c})$ near ∂M?
4. **Asymptotics of** g_+ near ∂M?

Geometry of conformal infinity

We return now to **conformally compact geometries** $(M, \mathbf{c}, I)$. Recall the **scale tractor** I is given $I = (\sigma, \nabla\sigma, -\frac{1}{d}(\Delta\sigma + J\sigma))$. We will consider in particular $(M, \mathbf{c}, I)$ which near the conformal infinity are **asymptotically of constant nonzero scalar curvature**. By imposing a constant dilation we may assume that I^2 approaches ± 1, **asymptotically hyperbolic/AdS** resp. **asymptotically de Sitter**.

The σ, equivalently scale tractor I, strongly links the geometry of $\Sigma = \mathcal{Z}(\sigma)$ to the ambient by a beautiful agreement of I and the normal tractor N:

Proposition

Let $(M^d, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$ and $I^2 = \pm 1 + \sigma^2 f$ for some smooth (weight -2) density f. Then Σ is a smoothly embedded hypersurface and, with N denoting the normal tractor for Σ, we have $N = I|_\Sigma$.

Proof.

For simplicity assume the case $I^2 = \pm 1$ (so $f = 0$ and the structure is ASC). As usual let us write $\sigma := h(X, I)$. Along $\mathcal{Z}(\sigma)$

$$I_A = \frac{1}{d} D_A \sigma \stackrel{g}{=} \begin{pmatrix} 0 \\ \nabla_a \sigma \\ -\frac{1}{d}\Delta\sigma \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{g}^{ab}(\nabla_a \sigma)\nabla_b \sigma = \pm 1$$

so $n_a := \nabla_a \sigma$ is the unit conormal and a computation gives $-\frac{1}{d}\Delta\sigma = -\frac{1}{d-1}\boldsymbol{g}^{ab} L^g_{ab} = -H^g$. □

Corollary

Let $(M^d, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma := \mathcal{Z}(\sigma) \neq \emptyset$, and that is **asymptotically Einstein** *in the sense that* $I^2|_\Sigma = \pm 1$, **and** $\nabla_a I_B = \sigma f_{aB}$ *for some smooth (weight -1) tractor valued 1-form f_{aB}. Then Σ is a* **totally umbilic hypersurface**.

Agreement of tractor connections

If we assume the stronger asymptotic: $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma^2 f_{aB}$, then along Σ, I_B is parallel to the given order, and so the tractor curvature satisfies

$$\kappa_{ab}{}^C{}_D I^D = \kappa_{ab}{}^C{}_D N^D = 0 \quad \text{along} \quad \Sigma.$$

This implies

$$\boxed{W_{ab}{}^c{}_d n^d = 0}, \quad \text{along} \quad \Sigma = \mathcal{Z}(\sigma)$$

$\therefore$ Fialkow $\mathcal{F}_{ab} = \frac{1}{n-2}(W_{acbd} n^c n^d + \mathring{K}^2_{ab} - \frac{|\mathring{K}|^2}{2(n-1)} \overline{\boldsymbol{g}}_{ab})$ vanishes, &

Theorem

Let $(M^{d\geq 4}, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma^2 f_{aB}$. Then the tractor connection of $(M, \mathbf{c})$ preserves the intrinsic tractor bundle of Σ, where the latter is viewed as a subbundle of the ambient tractors: $\mathcal{T}_\Sigma \subset \mathcal{T}$. Furthermore the restriction of the **parallel transport of $\nabla^{\mathcal{T}}$ coincides with the intrinsic tractor parallel transport of $\overline{\nabla}^{\overline{\mathcal{T}}}$**.

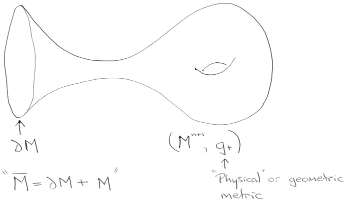
Summary to this point

An almost pseudo-Riemannian manifold with **non-zero generalised scalar curvature** has $\Sigma = \mathcal{Z}(\sigma)$ smoothly embedded.

Questions: E.g. $g = \sigma^{-2}\boldsymbol{g}$ – is **asymptotically Einstein** then:

1. **Asymptotics of** g near $\Sigma = \partial M$?:
   $I^2 = \pm 1 + \sigma f$ so g is asymptotically of constant scalar curvature and is resp. asymp. de Sitter/asyp. hyperbolic.
   $$R^g_{abcd} = \pm(g_{ac}g_{bd} - g_{ad}g_{bc}) + O(\sigma^{-3})$$
2. **Extrinsic geometry** of $(\partial M, \boldsymbol{c}|_{\partial M})$?:
   $$\mathring{K}_{ab} = 0, \quad \mathcal{F}_{ab} = 0, \cdots (see : arXiv : 2107.10381)$$
   Conformal geometry of $(M, \boldsymbol{c})$ near ∂M, e.g. $W_{ab}{}^{c}{}_{d}n^d = 0$.
3. **Intrinsic geometry** of $(\partial M, \boldsymbol{c}|_{\partial M})$?:
   For d odd, n even and $\nabla I = 0$ to high order (approx. σ^{d-1}) then
   $$0 = \overline{B}_{ab} = \overline{\Delta}^{n/2-2}\overline{\nabla\nabla W}_{acbd} + \text{lower order}$$
   the Fefferman-Graham obstruction tensor of $(\partial M, \boldsymbol{c}|_{\partial M})$

Part IV. Scattering of scalar fields in conformally compact mflds



Suppose on the interior one wants to solve

$$\left(\Delta^g + s(n-s)\frac{J^g}{d}\right)f = 0$$

where Δ^g is, as usual, the **wave operator** or metric **Laplacian** $g^{ab}\nabla_a\nabla_b$ for the conformally compact metric

$$g = g_+ = \sigma^{-2}\boldsymbol{g}$$

that is singular at the boundary ∂M. What are the right "Dirichlet" and "Neumann" boundary conditions? Mapping between these is one idea in scattering. Then s is the **spectral**

Differential operators by prolonged coupling

On an almost pseudo-Riemannian manifold $(M, \mathbf{c}, I)$ there is a canonical differential operator by **coupling** I^A **to** D_A , namely

$$I{\cdot}D := I^A D_A.$$

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight:

$$I{\cdot}D : \mathcal{E}^{\Phi}[w] \to \mathcal{E}^{\Phi}[w-1].$$

It will be useful to define define the *weight operator* $\mathbf{w}$: if $\beta \in \Gamma(\mathcal{B}[w_0])$ we have

$$\mathbf{w}\,\beta = w_0 \beta.$$

Then on $\mathcal{E}^{\Phi}[w]$ we have

$$I{\cdot}D \stackrel{g}{=} \begin{pmatrix} -\frac{1}{d}(\Delta\sigma + \mathrm{J}\sigma) & \nabla^a\sigma & \sigma \end{pmatrix} \begin{pmatrix} \mathbf{w}(d+2\mathbf{w}-2) \\ \nabla_a(d+2\mathbf{w}-2) \\ -(\Delta + \mathrm{J}\mathbf{w}) \end{pmatrix}.$$

$$= -\sigma\Delta + (d+2w-2)[(\nabla^a\sigma)\nabla_a - \frac{w}{d}(\Delta\sigma)] - \frac{2w}{d}(d+w-1)\sigma \mathrm{J}$$

The canonical degenerate Laplacian

Now on $M \setminus \mathcal{Z}(\sigma)$ in the metric $g_\pm = \sigma^{-2}\mathbf{g}$, with densities trivialised accordingly, we have

$$I{\cdot}D \stackrel{g_\pm}{=} -\Big(\Delta^{g_\pm} + \frac{2w(d+w-1)}{d}\mathrm{J}^{g_\pm}\Big).$$

In particular if $g_\pm$ satisfies $\mathrm{J}^{g_\pm} = \mp\frac{d}{2}$ (*i.e.* $\mathrm{Sc}^{g_\pm} = \mp d(d-1)$ or equivalently $I^2 = \pm 1$) then, relabeling $d+w-1 =: s$ and $d-1 =: n$, we have

$$\boxed{I{\cdot}D \stackrel{g_\pm}{=} -\big(\Delta^{g_\pm} \pm s(n-s)\big)}.$$

so solutions are **eigenvectors of the Laplacian** (and s is called the **spectral parameter**) as in **scattering theory**.

But on $\Sigma = \mathcal{Z}(\sigma) \neq \emptyset$, the conformal infinity, $I{\cdot}D$ degenerates and there the operator is first order. In particular if the structure is asymptotically ASC, in the sense that $I^2 = \pm 1 + \sigma^2 f$, for some smooth f, then along Σ

$$\boxed{I{\cdot}D = (d+2w-2)\delta_1}, \qquad \delta_1 \stackrel{g}{=} n^a\nabla^g_a - wH^g = \text{ conformal Robin}$$

Thus $I{\cdot}D$ is a **degenerate Laplacian**, natural to $(M, \mathbf{c}, I)$.

The $\mathfrak{sl}(2)$-algebra

$(M, \mathbf{c})$ be a conformal structure of dimension $d \geq 3$, $\sigma \in \Gamma(\mathcal{E}[1])$ and $I_A = \frac{1}{d} D_A \sigma$ (as usual). Then a direct computation gives

Lemma

Acting on any section of a weighted tractor bundle we have

$$[I \cdot D, \sigma] = I^2(d + 2\mathbf{w}),$$

where $\mathbf{w}$ *is the weight operator.*

Thus with **only the restriction that generalised scalar curvature is non-vanishing** we have:

Proposition (G.-Waldron)

Suppose that (M, c, σ) *is such that* I^2 *is nowhere vanishing. Setting* $x := \sigma$, $y := -\frac{1}{I^2} I \cdot D$, *and* $h := d + 2\mathbf{w}$ *we obtain the commutation relations*

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,$$

of **standard $\mathfrak{sl}(2)$-algebra generators**.

Application: (Extrinsic) Conformal Laplacian powers

Theorem

Let $(\overline{M}, \mathbf{c}, I)$ be conformally compact (and I^2 nowhere zero on $\Sigma = \partial M$). Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathcal{Z}(\sigma)$

$$P_k : \mathcal{E}^\Phi[\frac{k-n}{2}] \to \mathcal{E}^\Phi[\frac{-k-n}{2}] \quad \text{given by} \quad P_k := \Big(-\frac{1}{I^2} I{\cdot}D\Big)^k$$

is a tangential differential operator, and so determines a canonical differential operator $P_k : \mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \to \mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$. For k even this takes the form

$$P_k = \overline{\Delta}^{\frac{k}{2}} + \text{lower order terms.} \tag{11}$$

Proof.

From the $\mathfrak{sl}(2)$-identities we have $[x, y^k] = y^{k-1}k(h-k+1)$. Thus on $\mathcal{E}^\Phi[\frac{k-n}{2}]$

$$P_k(f + \sigma f_1) = y^k(f + x f_1) = P_k f + \sigma \widetilde{P}_k f_1.$$

So P_k is **tangential**. Expanding the $I{\cdot}D$s yields (11). □

Natural boundary problems

Suppose on a conformally compact manifold M_+ (with $M_+ \cup \partial M_+ = \overline{M}$) we wish to study solutions to

$$Pf := \left(\Delta^{g_+} + \frac{2w(d+w-1)}{d} \mathsf{J}^{g_+}\right) f = 0.$$

E.g. this is what is studied in the usual Poincaré-Einstein scalar scattering program.

Then one needs to fix suitable boundary conditions. E.g. in the case of Riemannian signature one wants some elliptic boundary problem. Since the boundary ∂M_+ is at infinity, with g_+ singular along ∂M_+, this is non-trivial.

But if we view f as the trivialisation of a density of weight w then $Pf \overset{g_+}{=} I{\cdot}Df$ and $I{\cdot}D$ is well-defined on all of $\overline{M}$ (and its smooth extension to M beyond ∂M_+). Thus it is natural to study the $I{\cdot}D$ problem. We do this **formally**.

First we treat an obvious Dirichlet-like problem where we view $f|_\Sigma$ as the initial data.

Asymptotic solutions of the first kind

Problem

Given $f|_\Sigma$, and an arbitrary extension f_0 of this to $\mathcal{E}^\Phi[w_0]$ over M, find $f_i \in \mathcal{E}^\Phi[w_0 - i]$ (over M), $i = 1, 2, \cdots$, so that

$$f^{(\ell)} := f_0 + \sigma f_1 + \sigma^2 f_2 + \cdots + O(\sigma^{\ell+1})$$

solves $I\cdot Df = O(\sigma^\ell)$, off Σ, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

$I\cdot Df = 0 \Leftrightarrow -\frac{1}{I^2} I\cdot Df = 0$ so we recast this via $\mathfrak{sl}(2) = \langle x, y, h\rangle$.

Set $h_0 = d + 2w_0$. By the identity $[x^k, y] = x^{k-1}k(h + k - 1)$:

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell+1)(h+\ell)f_{\ell+1} + O(x^{\ell+1}).$$

Now $hf_{\ell+1} = \big(h_0 - 2(\ell+1)\big)f_{\ell+1}$, thus

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell+1)(h_0 - \ell - 2)f_{\ell+1} + O(x^{\ell+1}). \qquad (12)$$

Induction: Suppose $yf^{(\ell)} = O(x^\ell)$, thus if $\boxed{\ell \neq h_0 - 2}$ we can solve $yf^{(\ell+1)} = O(x^{\ell+1})$ and this **uniquely determines** $f_{\ell+1}|_\Sigma$.

The obstruction on conformally compact manifolds

So we can solve to all orders provided we do not hit $\ell = h_0 - 2$ i.e. provided $w_0 \notin \{\frac{k-n}{2} \ : \ k \in \mathbb{Z}_{\geq 1}\}$. Otherwise (12) shows that

$$\ell = h_0 - 2 \quad \Rightarrow \quad yf^{(\ell)} = y\big(f^{(\ell)} + x^{\ell+1} f_{\ell+1}\big), \qquad \text{modulo } \ O(x^{\ell+1}),$$

regardless of $f_{\ell+1}$. It follows that the map $f_0 \mapsto x^{-\ell} yf^{(\ell)}$ is tangential and $x^{-\ell} yf^{(\ell)}|_\Sigma$ is the obstruction to solving $yf^{(\ell+1)} = O(x^{\ell+1})$. Then by a simple induction this is seen to be a non-zero multiple of $y^{\ell+1} f_0|_\Sigma$:

Proposition

If $\ell = h_0 - 2$ then the smooth extension is (in general) obstructed by $P_{\ell+1} f_0|_\Sigma$, where $P_{\ell+1} = (-\frac{1}{I^2} I \cdot Df)^{\ell+1}$ is a tangential operator on densities of weight w_0.

If $\ell = h_0 - 2$ then the extension can be continued with **log terms**.
If $\overline{M}$ is **almost Einstein** to sufficiently high order then:

- the **odd order** $P_{\ell+1}$ **vanish identically**; and
- the **even order** $P_{\ell+1}$ are the **GJMS operators** on $(\partial M_+, \bar{\mathbf{c}})$.

(Formal) solutions of the second kind

Now we consider the more general type of solution:

Problem

Given $\overline{f}_0|_\Sigma \in \Gamma\mathcal{E}^\Phi[w_0-\alpha]|_\Sigma$ *and an arbitrary extension* $\overline{f}_0$ *of this to* $\Gamma\mathcal{E}^\Phi[w_0-\alpha]$ *over* $\overline{M}$*, find* $\overline{f}_i \in \mathcal{E}^\Phi[w_0-\alpha-i]$ *(over* $\overline{M}$*),* $i = 1, 2, \cdots,$ *so that*

$$\overline{f} := \sigma^\alpha(\overline{f}_0 + \sigma\,\overline{f}_1 + \sigma^2\,\overline{f}_2 + \cdots + O(\sigma^{\ell+1})) \tag{13}$$

solves $I \cdot D\overline{f} = O(\sigma^{\ell+\alpha})$*, off* ∂M_+*, for* $\ell \in \mathbb{N} \cup \infty$ *as high as possible.*

Now, if α is not integral, this Problem is outside the realm of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and its modules. But it is straightforward to show that for any $\alpha \in \mathbb{R}$:

$$[x^\alpha, y] = x^{\alpha-1}\alpha(h+\alpha-1). \tag{14}$$

It follows immediately from (14) that $I \cdot D\overline{f} = 0$ has:

- no solution if $\alpha \notin \{0, h_0 - 1\}$, where $h\overline{f} = h_0\overline{f}$; and
- if $\alpha = h_0 - 1$ and $\overline{f} = \sigma^\alpha f$ then

$$I{\cdot}D\overline{f} = \sigma^\alpha I{\cdot}Df$$

So $\overline{f}$ is a solution iff f is!

So in this way **second solutions** arise from **first** and vv.

For $w_0 \notin \{\frac{k-n}{2} \ : \ k \in \mathbb{Z}_{\geq 1}\}$, and writing $F = f$, $G = \sigma^{-\alpha}\overline{f}$ we can combine these to a general solution

$$F + \sigma^{h_0 - 1} G = F + \sigma^{n+2w_0} G$$

or, trivialising the densities on M_+ using the generalised scale σ:

$$f = \sigma^{n-s} F + \sigma^s G = \sigma^{-w_0}(F + \sigma^{h_0-1} G)$$

where $s := w_0 + n$. Which is the form of solution used in the **scattering theory** (of Graham-Zworski, Mazzeo-Melrose, $\cdots$). The $\mathfrak{sl}(2)$ approach above solves the asymptotics of F and G.

For $w_0 \in \{\frac{k-n}{2} : \ k \in \mathbb{Z}_{\geq 1}\}$ we need log terms . . .It works.

The END of lecture two

Plan of Lecture Three

Part I. Hypersurfaces in conformal geometry – second taste.
The singular Yamabe problem.

Part II. Holography and higher fundamental forms

Part III Asymptotics of asymptotically de Sitter spacetimes

Part IV. Projective compactification

Part I. A holographic approach to submanifolds

Holographic means that we use a boundary problem to find invariants!

Part I. Singular Yamabe problem and higher Willmore invariants

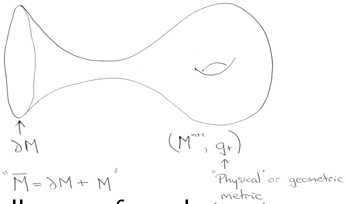
* G-. + Waldron, Conformal hypersurface geometry via a boundary Loewner-Nirenberg-Yamabe problem. *Comm. Anal. Geom.* (2021),
* G-. + Waldron, Andrew Renormalized volume. *Comm. Math. Phys.* (2017)
* Arias, G-. , Waldron, Conformal geometry of embedded manifolds with boundary from universal holographic formuae. *Adv. Math.* (2021),

Part II: Higher conformal fundamental forms

Blitz, G-. , Waldron, Conformal Fundamental Forms and the Asymptotically Poincaré–Einstein Condition, Indiana - in press.

A singular Yamabe problem

Recall a **conformal compactification** of a Riemannian manifold (M^{n+1}, g_+) is a manifold $\overline{M}$ with boundary ∂M s.t.:
- $\exists\ \overline{g}$ on $\overline{M}$, with $g_+ = r^{-2}\overline{g}$, where
- r a **defining function** for ∂M: $\partial M = \mathcal{Z}(r)$ & $dr_p \neq 0\ \forall p \in \partial M$.



$\Rightarrow$ canonically a conformal structure on boundary: $(\partial M, [\overline{g}|_{\partial M}])$.

Question/variant: Given $\overline{g}$ (or really $\mathbf{c} = [\overline{g}]$) can we find a defining function $r \in C^\infty(\overline{M})$ for $\Sigma = \partial M$ s.t.

$$\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1)?$$ NB: This satisfied for the Poincaré Ball

cf. Loewner-Nirenberg, Aviles and McOwen – related interior problems.

The obstruction density of ACF

Can we solve $\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1)$? formally (i.e. power series) along the boundary? **Answer: No** - in general can get:

Theorem (Andersson, Chruściel, & Friedrich)

$$\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1) + r^{n+1}\mathcal{B}_n.$$

Furthermore (they show)

$$\mathcal{B}_2 = \delta \cdot \delta \cdot \mathring{L} + \textit{lower order}$$

is a conformal invariant of $\Sigma^2 = \partial M$ *(and more).*

Theorem.[G. + Waldron] For $n \geq 2$ $\mathcal{B}_n$ is a conformal invariant of $\Sigma = \partial M$, and $\mathcal{B}_2 =$ **Willmore Invariant** $= \overline{\Delta} H +$ lower order!

- For n even the invariant $\mathcal{B}_n$ is **higher order analogue** of $\mathcal{B}_2 = \mathcal{B}$.

Recall a conformal manifold has a canonical conformal metric $\boldsymbol{g} \in S^2T^*M[2]$, and a metric $g_+ \in \mathbf{c}$ is equivalent to a **scale**:

$$\boxed{g_+ = \sigma^{-2}\boldsymbol{g}} \quad \Leftrightarrow \quad \boxed{\sigma \in \Gamma(\mathcal{E}_+[1]).}$$

Via the Thomas-D operator $\mathrm{D} = \frac{1}{d}D$ the scale is equivalent to the

$$\boxed{\textbf{scale tractor} \quad I_A := \mathrm{D}_A\sigma,} \qquad \text{and}$$

Lemma

$\mathrm{Sc}(g_+) = -n(n+1) \Leftrightarrow I^2 := h(I,I) = 1$

So we come to a "conformal Eikonal equation" $(\mathrm{D}_A\sigma)(\mathrm{D}^A\sigma) = 1$, where σ a **defining density** for Σ. **NB:**

- If $\boxed{\text{we could solve uniquely then } \Sigma \hookrightarrow (M,\mathbf{c}) \textbf{ determines } g \in \mathbf{c}}$.

Then invariants of conf. compact (M, g_+) would be invariants of Σ.

Thus to solve the singular Yamabe problem formally we come to the following **non-linear** problem:

Problem: For a conformal manifold $(M, \mathbf{c})$ and an embedding $\iota : \Sigma \to M$ solve

$$I_A I^A = (\mathrm{D}_A \sigma)(\mathrm{D}^A \sigma) = 1 + O(\sigma^\ell)$$

for ℓ as high as possible, and σ a Σ defining density.

• Now we use again the result – that there is an $\mathfrak{sl}(2)$ generated by $x := \sigma$, $y := -\frac{1}{I^2} I^A D_A$:

In particular from the standard $\mathfrak{sl}(2)$ identities, we have

$$[I \cdot D, \sigma^{k+1}] = I^2 \sigma^k (k+1)(d + k + 2\boldsymbol{w}).$$

This allows an inductive solution (using also **other tractor identities**):

Lemma

Suppose that $\underline{\sigma \in \Gamma(\mathcal{E}[1]) \text{ defines } \Sigma = \partial M_+ \text{ in } (\overline{M}, \mathbf{c})}$ *and*

$$I_\sigma^2 = 1 + \sigma^k A_k \quad \text{where} \quad A_k \in \Gamma(\mathcal{E}[-k])$$

is smooth on M, *and* $k \geq 1$, *then*

- *if* $k \neq (n+1)$ *then* $\exists\ f_k \in \Gamma(\mathcal{E}[-k])$ *s.t.* $\sigma' := \sigma + \sigma^{k+1} f_k$ *satisfies* $I_{\sigma'}^2 = 1 + \sigma^{k+1} A_{k+1}$, *where* A_{k+1} *smooth;*
- *if* $k = (n+1)$ *then:* $I_{\sigma'}^2 = I_\sigma^2 + O(\sigma^{n+2})$.

Proof.

Squaring with the tractor metric, using the $\mathfrak{sl}(2)$, etc

$$\begin{aligned}(\mathrm{D}\sigma')^2 &= (\mathrm{D}\sigma + \mathrm{D}(\sigma^{k+1} f_k))^2 \\ &= I_\sigma^2 + \frac{2}{n+1} I_\sigma \cdot D(\sigma^{k+1} f_k) + (\mathrm{D}(\sigma^{k+1} f_k))^2 \\ &= 1 + \sigma^k A_k + \frac{2\sigma^k}{n+1}(k+1)(n+1-k) f_k + O(\sigma^{k+1}).\end{aligned}$$

□

The distinguished defining density and obstruction density

Theorem (G.-, Waldron, CAG '21, Trends Math. '15)

For Σ^n embedded in $(M^{n+1}, \mathbf{c})$ there is a distinguished defining density $\bar{\sigma}$, **unique** *modulo $+O(\sigma^{n+2})$, s.t.*

$$I_{\bar{\sigma}}^2 = 1 + \bar{\sigma}^{n+1}\mathcal{B}_{\bar{\sigma}}.$$

Moreover:

$$\mathcal{B} := \mathcal{B}_{\bar{\sigma}}|_\Sigma \in \Gamma(\mathcal{E}_\Sigma[-n-1])$$

is determined by $(M, \mathbf{c}, \Sigma)$ and is a **natural conformal invariant***.*

For n even $\mathcal{B} = 0$ **generalises the Willmore equation** *in that:*

$$\mathcal{B} = \bar{\Delta}^{\frac{n}{2}} H + \text{lower order terms};$$

while for n odd $\mathcal{B}$ has no linear leading term.

Corollary (ACF + above implies)

On (M, g), if there is a sign changing smooth solution of sing. Yamabe*: $|du|^2 - \frac{2}{n+1} u \left(\Delta^g + \frac{\mathrm{Sc}^g}{2n}\right) u = 1$ then $\Sigma := \mathcal{Z}(u)$ is a higher Willmore hypersurface – i.e. it satisfies $\mathcal{B} = 0$.

$$\star\ \mathrm{Sc}^{u^{-2}g} = -n(n+1) \text{ eqn}$$

$\mathcal{B}$ is variational

For suitable regularisations $\overline{M}_\epsilon$ of conformally compact manifolds $\overline{M}$:

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}_\epsilon} \sqrt{g_+} = \frac{v_n}{\epsilon^n} + \cdots + \frac{v_1}{\epsilon} + \mathcal{A} \log \epsilon + V_{ren} + O(\epsilon).$$

Theorem (Graham PAMS 2017. G.-+Waldron CMP 2017)

If $g_+ = \bar{\sigma}^{-2}\mathbf{g}$, where $\bar{\sigma}$ an approximate solution of the sing. Yamabe problem then $\mathcal{A}$ a conformal invariant of $\Sigma \hookrightarrow M$ and

$$\frac{\delta \mathcal{A}}{\delta \Sigma} = \frac{(n+1)(n-1)}{2}\mathcal{B}$$

So the anomaly term in the renormalised volume expansion provides an **energy** with **functional gradient the obstruction density**, in other words $\mathcal{A}$ is an energy generalising the Willmore energy.

Further Invariants by conformal holography.

Recall the Singular Yamabe Thm:

Theorem (SY Thm G.-, Waldron arXiv:1506.02723 = CAG '21)

For Σ^n embedded in $(M^{n+1}, \mathbf{c})$ there is a distinguished defining density $\bar{\sigma}$, **unique** *modulo $+O(\sigma^{n+2})$, s.t.*

$$I_{\bar{\sigma}}^2 = 1 + \bar{\sigma}^{n+1}\mathcal{B}_{\bar{\sigma}}.$$

Moreover:

$$\mathcal{B} := \mathcal{B}_{\bar{\sigma}}|_{\Sigma} \in \Gamma(\mathcal{E}_{\Sigma}[-n-1])$$

is a natural invariant $\cdots$
Etcetera

Corollary (above implies)

A $(\overline{M}, \mathbf{c})$ has a **canonical** *conformally compact structure up to $+O(\sigma^d)$.*

Part II. All hypersurface invariants via holography?

The construction can be used to obtain other hypersurface invariants: Our Theorem above shows that:

$$(M, \mathbf{c}, \Sigma) \quad \text{determines} \quad \bar{\sigma} \quad \text{modulo} + O(\sigma^{n+2}).$$

Suppose that $\mathcal{I}$ is any coupled conformal invariant of $(M, \mathbf{c}, \bar{\sigma})$ involving only the jet $j^{n+1}\bar{\sigma}$. Then along Σ

$$\boxed{\mathcal{I}\big|_{\Sigma} \quad \text{is a conformal invariant of} \quad (M, \mathbf{c}, \Sigma).}$$

This **holographic** approach fails at order $n+2$ because of the existence of the **obstruction invariant** $\mathcal{B}$ and ambiguity. This is an analogue of the use Fefferman-Graham's Poincaré and ambient metric constructions to find conformal invariants – that fails at order $n+1$ because of **Bach** B_{ab} in dimension 4 and the **Fefferman-Graham obstruction tensor** in higher even dimensions.

The obstructions to Poincaré-Einstein (PE)

Conformally compact manifolds are often assumed to be (asymptotically) Poincaré-Einstein, meaning that

$$\text{Trace-free}(Ric^{g_+}) = 0.$$

But what does it mean?

Given a conformal manifold with boundary $(\overline{M}, \mathbf{c})$ *does it admit a smooth* **asymptotically** *PE metric with* ∂M *the conformal infinity?*

It turns out that the trace free second fundamental form $\mathring{\mathrm{II}}$ is an obstruction. At the next order the **tf Fialkow tensor** is a next obstruction:

$$\mathring{\mathcal{F}}_{ab} = \frac{1}{d-3}(W_{acbd}n^c n^d + \mathring{II}^2_{(ab)_0})$$

How do we systematically find the higher order obstructions?

The almost Einstein tensor E_{ab}

In a PE manifold $(\overline{M} = M \cup \partial M, g^o)$ the Schouten tensor of the metric satisfies

$$P^{g^o}_{ab} = \lambda g^o_{ab} \qquad \text{on} \quad M \tag{15}$$

But both g^o and P^{g^o} are singular at ∂M. HOWEVER given $\sigma \in \Gamma(\mathcal{E}[1])$ the quantity

$$\text{Trace-Free}(\nabla^g_a \nabla^g_b \sigma + \sigma P^g_{ab}) \qquad g \in \mathbf{c},$$

is conformally invariant and smooth on $\overline{M}$. By the SY Thm, if: 1. $Z(\sigma) = \partial M$, 2. $I^2 = \pm 1 + \sigma^d B$ (B smooth), then

$$\boxed{E_{ab} := \text{Trace-Free}(\nabla^g_a \nabla^g_b \sigma + \sigma P^g_{ab})} \qquad \text{is determined by } (\overline{M}, \mathbf{c})$$

up to $+O(\sigma^n)$. On the interior $\boxed{E_{ab} = \sigma P^{g^o}_{(ab)_0}}$ where $g^o := \boldsymbol{g}/\sigma^2$.

So E_{ab} extends $\sigma P^{g^o}_{ab}$ smoothly to the boundary $\therefore$

$$\boxed{E_{ab} \equiv 0 \quad \text{iff} \quad g^o \text{ is a PE metric}}$$

Summarising:

- $E_{ab} :=$ Trace-Free$(\nabla^g_a \nabla^g_b \sigma + \sigma P^g_{ab})$ extends $\sigma P^{g^o}_{ab}$ smoothly to the boundary.
- E_{ab} vanishes iff g^o is a PE metric.
- E_{ab} depends only on the conformal embedding $\partial M \hookrightarrow (\overline{M}, \mathbf{c})$, up to $+O(\sigma^n)$. Thus

Lemma: "The jets of E_{ab} along ∂M are extrinsic hypersurface invariants that obstruct the existence of PE metrics in $\mathbf{c}|_M$".

E.g. zero jet:

Proposition: $E_{ab}|_{\partial M} = \mathring{\rm II}_{ab}$.

Proof: 1. σ SY means $I_A|_{\partial M} = N_A$ and $n_a := \nabla_a \sigma$ is a weight 1 unit conormal. Then 2. differentiating $I^2 = 1 + O(\sigma^{n+1})$ gives

$$N^B \nabla_a I_B = 0 \quad \& \quad n^b \nabla_b I_A = 0 \quad \text{mod } \langle X_A \rangle \qquad \text{along } \partial M.$$

So, mod $\langle X_A \rangle$,

$$\begin{pmatrix} 0 \\ E_{ab} \\ * \end{pmatrix} = \nabla_a I_B \overset{\partial M}{=} \underline{\nabla}_a N_B \overset{\text{see p6 Prop}}{=} \begin{pmatrix} 0 \\ \mathring{\rm II}_{ab} \\ * \end{pmatrix}$$

Higher Fundamental Forms

So $\boxed{\mathring{\mathrm{II}} \text{ is an obstruction to Poincaré-Einstein (PE).}}$

Next, recall the Cherrier-Robin operator:

$$\delta_1 \overset{g}{:=} n^a \nabla_a - wH^g : \Gamma(\mathcal{T}^\Phi[w]) \to \Gamma(\mathcal{T}^\Phi[w-1]|_{\partial M})$$

where $\mathcal{T}^\Phi[w]$ means any weight w tractor bundle – or simply densities of that weight.
There's a version for rank 2 trace-free symmetric tensors of weight w. And

$$\delta_1 E_{ab} = n^c n^d W_{cabd} - \mathring{\mathrm{II}}^2_{(ab)\circ} = -(d-3)\mathring{\mathcal{F}}_{ab} \in \Gamma(S^2_o T^* \partial M)$$

where $\mathcal{F}$ is the Fialkow tensor and $\mathring{\mathrm{II}}^2$ is the obvious composition of II_0 with itself. Note that $J^1_{\partial M} E_{ab}$ is captured by the two extrinsic invariants $\mathring{\mathrm{II}}$ (which gets $J^0_{\partial M} E$) and $\delta_1 E_{ab}$. So

$\boxed{\mathring{\mathrm{III}} := \delta_1 E_{ab} \text{ is an obstruction to PE.}}$

Making higher fundamental forms

To make higher order analogues of $\mathring{\mathrm{II}}$ and $\mathring{\mathrm{III}}$ we need higher analogues of the operator δ_1.

STEP 1: We want E in a tractor quantity.

$$P_{AB} := \mathrm{D}_A I_B = \mathrm{D}_A \mathrm{D}_B \sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_{ab} & * \\ 0 & * & * \end{pmatrix}$$

does this.

STEP 2: We can then form e.g.

$$\delta_1 (I \cdot D)^{K-1} P_{AB} \quad \text{or better} \quad \delta_K P_{AB}$$

where δ_K are the conformal higher Neumann operators.

STEP 3: Actually STEP 2 needs a lot of refining to extract a symmetric trace-free tensor again . . . see Blitz, G. Waldron arXiv:2107.10381

The Punchline

Theorem BGW: Let $d \geq 3$ and let $2 \leq K < \frac{d+3}{2}$. For any embedded hypersurface Σ in a conformal d manifold there is a well-defined *canonical Kth fundamental form* $\underline{\overline{\mathring{\mathrm{K}}}}$ is defined by

$$\underline{\overline{\mathring{\mathrm{K}}}} := \delta_{(K-2)} E.$$

• Each K^{th}-fundamental form is an extrinsic hypersurface conformal invariant that depends, along Σ, on $K-1$ transverse derivatives of the ambient conformal structure $\mathbf{c}$.

• Each $\underline{\overline{\mathring{\mathrm{K}}}}$ is an obstruction to the existence of an asymptotically PE $g_+ \in \mathbf{c}$.

Next:

• If $\mathring{\mathrm{II}}$, $\mathring{\mathrm{III}}$, $\cdots$, $\underline{\overline{\mathring{\lfloor \frac{d+3}{2} \rfloor}}}$ vanish then we can define higher fundamental forms to $K = n = d - 1$ and :

Theorem BGW If $\mathring{\mathrm{II}}$, $\mathring{\mathrm{III}}$, $\cdots$ $\underline{\overline{\mathring{d-1}}}$ vanish, then

$$g_+ = \boldsymbol{g}/\sigma^2$$

is asymptotically PE meaning $E = O(\sigma^{n-1})$

Let $(\widetilde{M}^4, \widetilde{g}_{ab})$ be a spacetime (meaning $\tilde{g}$ Lorentzian) that satisfies the Einstein field equations with **positive cosmological constant** $\Lambda = 12\lambda$,

$$\widetilde{R}_{ab} - \frac{1}{2}\widetilde{R}\widetilde{g}_{ab} + \Lambda\widetilde{g}_{ab} = \widetilde{T}_{ab}, \qquad (16)$$

where $\widetilde{T}_{ab}$ is the stress-energy tensor.

Definition

$(\widetilde{M}, \widetilde{g}_{ab})$ is asymptotically de Sitter if there exists a manifold M, with boundary Σ and a metric g_{ab}, such that

- there is an embedding $\varphi : \widetilde{M} \to M$ such that $\varphi(\widetilde{M}) = M \setminus \Sigma$,
- the metric $\mathrm{g}_{ab} \in \mathrm{c}$ is regular on M and satisfies $\widetilde{g}_{ab} = \Omega^{-2}\mathrm{g}_{ab}$ (on $\widetilde{M}$) for some smooth non-negative function Ω on M,
- Ω is a defining function of the boundary Σ, i.e. $\Sigma = \Omega^{-1}(0)$ and $d\Omega$ is nowhere zero on Σ,
- the stress-energy tensor $\widetilde{T}_{ab}$ vanishes along Σ.

Asymptotically de Sitter via conformal/tractor picture

So the structure is a conformally compact(ified) manifold $(\overline{M}, \mathbf{c}, \tilde{\sigma})$, with $\overline{M} = M$ and

$$\tilde{g} = \tilde{\sigma}^{-2}\boldsymbol{g}$$

where $\tilde{\sigma}$ is represented by Ω with respect to the scale $g \in \mathbf{c}$. $\Sigma = \mathcal{Z}(\tilde{\sigma})$ and there

$$\tilde{I}^A \tilde{I}_a = -\lambda < 0 \quad \text{where} \quad \tilde{I}_A := \frac{1}{d} D_A \tilde{\sigma}.$$

So, at all points of $\Sigma := \mathcal{Z}(\tilde{\sigma})$,

$$\boldsymbol{g}^{-1}(\nabla\tilde{\sigma}, \nabla\tilde{\sigma}) < 0 \quad \Rightarrow \quad \Sigma \text{ spacelike.}$$

So if $\tilde{M}$ complete (and only asymp. de Sitter) we would have $\Sigma = \Sigma_- \cup \Sigma_+$. We will work on just Σ_+ and view that as Σ – say following an initial singularity/big bang.

One can consider different asymptotic behaviours for the stress-energy $\tilde{T}_{ab}$. We set

$$\tilde{T}_{ab} = \tilde{\sigma}^q \tau_{ab} \quad \text{and considered} \quad q = 0, 1, 2,$$

where τ_{ab}, with a suitable weight, is smooth to Σ. Then the (trace-free) Einstein equations become

$$E_{ab} = \frac{\widetilde{\sigma}^{q+1}}{2} \mathring{\tau}_{ab}, \tag{17}$$

and, as earlier $E_{ab} = trace - free(\nabla_a \nabla_b \tilde{\sigma} + \tilde{\sigma} P_{ab})$.

Definition

Let $i \in \{0, 1, 2, 3, 4\}$. A conformal fundamental form $\mathring{\mathcal{K}}_{ab}^{(i+2)}$ can be defined as

$$\mathring{\mathcal{K}}_{ab}^{(i+2)} := \begin{cases} \mathring{K}_{ab} & \text{for} \quad i = 0, \\ \delta_R \circ (\mathrm{ID})^{i-1} E_{ab} & \text{for} \quad 1 \leq i \leq 4. \end{cases} \tag{18}$$

In the above e.g.

$$\mathrm{ID} : \mathcal{E}_{(ab)_0}[w] \to \mathcal{E}_{(ab)_0}[w-1] \quad \text{for} \quad w \neq -2, 3.$$

is defined by

$$\mathrm{ID} t_{ab} := p^* \circ r \circ I^A_{\tilde{\sigma}} D_A \circ p\,(t_{ab})\,, \qquad p \text{ maps into tractors}$$

or

$$\begin{aligned}\mathrm{ID} t_{ab} =& 2(w-1)\sqrt{\lambda}\Big(\left[\nabla_n + (w-2)\rho\right] t_{ab} - \tfrac{2(w-2)}{(w-3)(w+2)} n_{(a}\nabla \cdot t_{b)_0} \\ &+ \frac{2}{w-3}\left[n \cdot \nabla_{(a} t_{b)_0} + t_{(a} \cdot \nabla_{b)_0} n\right] \Big) \\ &- \tilde{\sigma}\Big(\Delta t_{ab} + (w-2) J t_{ab} + \tfrac{8}{(w-3)(w+2)} \nabla_{(a}\nabla \cdot t_{b)_0} - 4 P_{(a} \cdot t_{b)_0} \Big)\end{aligned}$$

The earlier treatment defined fundamental forms via a scale σ, defining Σ, such that $I^2 = \pm 1 + \sigma^d B$. Here $\tilde{I}_A$ satisfies

$$\tilde{I}^2 = -\lambda + \frac{1}{12}\tilde{\sigma}^{q+2}\tau \quad \tau = \boldsymbol{g}^{ab}\tau_{ab},$$

so we restrict to treating the conformal fundamental forms up to $\mathring{\mathcal{K}}^{(q+4)}_{ab}$ for the given value of the decay parameter q.

In terms of the metric g, τ_{ab} is represented by the "unphysical" Stress energy T_{ab} and $\tilde{T}_{ab} = \Omega^q T_{ab}$. Then:

Theorem: On Σ: For $q = 0, 2$ we have $\mathring{\mathcal{K}}^k_{ab} = 0$ for $2 \le k \le q+2$ and

$$\mathring{\mathcal{K}}^{(q+3)}_{ab} = C(q, \Lambda) \, \mathring{\top} \, (\mathrm{T}_{ab}) \quad \text{on} \quad \Sigma,$$

where $(\mathring{\top})$ denotes the (trace-free) projection on Σ. If $q \ge 1$:

$$\overline{\nabla}^b \mathring{\mathcal{K}}^{(4)}_{ab} = \begin{cases} \left(\frac{\Lambda}{3}\right)^{3/2} \left(\frac{1}{3} \overline{\nabla}_a \mathrm{T} - \sqrt{\frac{\Lambda}{3}} j_a \right) & \text{for} \quad q = 1, \\ -\frac{\Lambda^2}{9} \top \left(n^b \mathrm{T}_{ab} \right) & \text{for} \quad q = 2 \end{cases}$$

on Σ, where $\overline{\nabla}_a$ is induced Levi-Civita on Σ and $\mathrm{T}_{ab} n^b = \Omega j_a + \mathcal{O}(\Omega)$. The $\mathring{\mathcal{K}}^{(4)}_{ab}$ is otherwise undetermined by the local data on the conformal boundary Σ but a constraint

$$\mathrm{T}_{ab} = -n_a n_b \mathrm{T} \quad \text{on} \quad \Sigma \quad \text{for} \quad q = 1 \tag{19}$$

arises.

There are constraints on $T_{ab} \cdots$

Clarify

Some formulae

$$\mathring{\mathcal{K}}^{(3)}_{ab} := \delta_R \left(\sqrt{\lambda} \nabla_a n_b + \widetilde{\sigma} P_{ab} - \frac{1}{4} g_{ab} \left(\sqrt{\lambda} \nabla_a n^a + \widetilde{\sigma} J \right) \right).$$

Gives

$$\mathring{\mathcal{K}}^{(3)}_{ab} := \sqrt{\lambda} \left(C^{\top}_{nabn} - \mathring{K}_{bc} \mathring{K}_a{}^c + \frac{1}{3} \bar{g}_{ab} \mathring{K}_{cd} \mathring{K}^{cd} \right),$$

Fine print. Computed as above, the fourth fundamental form is

$$-2\lambda \mathring{\top} (\mathring{K}^{cd} C_{acbd})$$

so is trivially zero when $\mathring{K}^{cd}$ is. That's because in dimension 4 this where the "non-local" term arisies. But

$$\mathring{\mathcal{K}}^{(4)}_{ab} := \lambda \left(\overline{\nabla}^c C^{\top}_{n(ab)c} - A^{\top}_{(a|n|b)} - \frac{1}{3} H C_{nanb} \right).$$

is a **conformal invariant** that captures this. This is the part **not determined** by the local formal expansion of the Einstein eqns.

Finally: $\mathring{\mathcal{K}}^{(5)}_{ab} := \delta_R \circ \mathrm{ID}^2 E_{ab} = 6\lambda^{\frac{3}{2}} \mathring{\top} (B_{ab}) \quad \text{for} \quad q \geq 1,$

Summary

- $q = 0$

$$C_{nanb} \stackrel{\Sigma}{=} \frac{1}{2} \mathring{\top}(\tau_{ab}) \,,$$

- $q = 1$

$$C_{abcd} \stackrel{\Sigma}{=} 0, \quad \tau_{ab} \stackrel{\Sigma}{=} -n_a n_b \tau,$$

$$\overline{\nabla}^b A^\top_{anb} \stackrel{\Sigma}{=} \lambda j_a - \frac{\sqrt{\lambda}}{3} \overline{\nabla}_a \tau,$$

$$\mathring{\top}(B_{ab}) \stackrel{\Sigma}{=} \frac{\sqrt{\lambda}}{3} \left(9\nabla_n \tau^\top_{ab} - 3\bar{g}_{ab} \nabla_n \tau_{nn} + 8H\tau^\top_{ab} - H\bar{g}_{ab}\tau_{nn} \right),$$

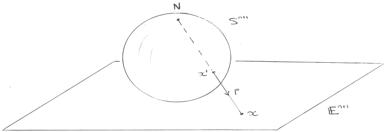
$$B_{nn} \stackrel{\Sigma}{=} 0, \quad B^\top_{an} \stackrel{\Sigma}{=} \sqrt{\lambda} \left(\nabla_n \tau^\top_{an} + \frac{1}{3} \overline{\nabla}_a \tau \right),$$

- $q = 2$

$$C_{abcd} \stackrel{\Sigma}{=} 0, \quad \overline{\nabla}^b A^\top_{anb} \stackrel{\Sigma}{=} \lambda \tau^\top_{an},$$

$$\mathring{\top}(B_{ab} + 3\lambda\tau_{ab}) \stackrel{\Sigma}{=} 0, \quad B_{nn} \stackrel{\Sigma}{=} 0, \quad B^\top_{an} \stackrel{\Sigma}{=} -\lambda \tau^\top_{an}.$$

Note **Stereographic projection**



is not a very good compactification for Euclidean scattering, or for the representation theory of the Euclidean group. As the "one point" has little room for information.

We have used conformal geometry to understand the compactification of space-times and more generally pseudo-Riemannian manifolds, and also some related problems. Are there other similar tools based around geometries other than conformal? Let's first revisit our conformal theory.

Recall conformal compact'n of $\mathbb{H}^{n+1}$ – the Poincaré ball

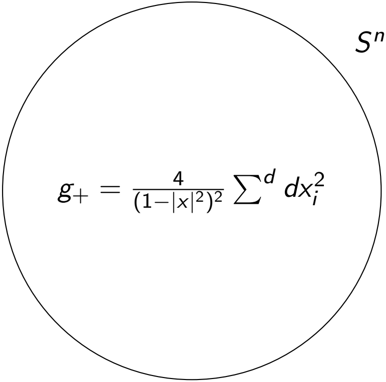
Escher's circle limit

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 + \partial\mathbb{H}^2$$

The embedding gives the compactification

$\mathbb{H}^d$ embedded conformally in Euclidean $\mathbb{E}^d$ – Poincaré-Ball

$S^n = \partial\mathbb{H}^{n+1}$



$$g_+ = \frac{4}{(1-|x|^2)^2}\sum^d dx_i^2$$

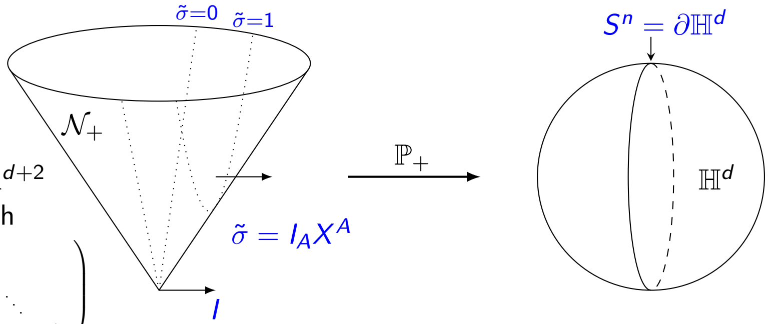
$$\overline{\mathbb{H}^d} = \mathbb{H}^d + \partial\mathbb{H}^d$$

Poincaré compactification via $\mathbb{P}_+(\mathit{nullcone})$

Conformal compactification of $\mathbb{H}^d$ by **symmetry breaking**:

i.e. $\mathbb{R}^{d+2}$ with

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$



$S^d = \mathbb{P}_+(\mathcal{N}_+ \subset \mathbb{R}^{d+2} \setminus \{0\})$ is model of flat conformal geometry.
$G := SO_o(d+1,1)$ acts transitively. $I \in \mathbb{R}^{d+2}$, spacelike $h(I,I) = 1$
Symmetry reduction by I: $\Rightarrow H = SO_o(d,1)$ orbits. Right hemi. is conformal compactification $\overline{M}_c$ of $\mathbb{H}^d$

Minkowski, de Sitter, and AdS

Recall that the Lorentzian conformally flat model is $(S^1 \times S^n, \boldsymbol{c})$ arising as the ray projectivisation

$$S^1 \times S^n = \mathbb{P}_+(\{x_0^2 + x_1^2 = x_2^2 + \cdots x_{d+2}^2\}) \quad \leftarrow G = S0_o(d,2) \text{acts trans}^{\text{y}}$$

of the null quadric in

$$\mathbb{R}^{d+2} \quad \text{with bilinear form} \quad h = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \boxed{d := n+1}$$

Then $I_A \in \mathbb{R}^{d+2}$ – constant $\Leftrightarrow$ **parallel tractor** with:
• $I^2 = -1$: Two copies of **conf. compactified de Sitter** on *interval* $\times S^n$; $H = S0_o(d,1)$ orbits
• $I^2 = 0$: Two copies of **conf. compactified Minkowski**, as in Ein. cyl.; $H =$Poincaré grp orbits • $I^2 = 1$: Two copies of **c.c. anti-de Sitter** on $S^1 \times$ *hemisphere*; $H = S0_o(d-1,2)$ orbits

Bigger groups: H vs G and Orbit Decompositions

In each example above there is implicitly a larger group $G \supset H$:

• **Poincaré ball and compactifying boundary** arise as two $H = SO_+(d,1)$ orbits on $S^d = G/P$ where

$$\boxed{G = S0_+(d+1,1) \text{ and } P \text{ maximal parabolic in } G}.$$

The larger homogeneous space G/P encodes how the orbits **smoothly** fit together – i.e. the conformal compactification.

Similarly:

• **Stereographic (conformal) compactification** of $\mathbb{E}^{n+1}$ arises as two $H =$ Euclidean group orbits on $S^{n+1} = G/P$ – with same G and P. I.e. $\boxed{\text{Stereo. encoded by } H \hookrightarrow G}$

• etc

Curving homogeneous spaces

For a Lie group G and closed Lie group P, homogeneous spaces G/P are **geometries** in the sense of **Klein**. There are often canonical curved generalisations:

Theorem (Cartan, Tanaka, $\cdots$)

If P is a parabolic subgroup of a semisimple Lie group G then there is a **canonical** *notion of geometry*

$$\begin{array}{ccc} \mathcal{G} & \leftarrow & P \\ \downarrow & & \\ M & & \end{array} \quad \textit{modelled on} \quad \begin{array}{ccc} G & \leftarrow & P \\ \downarrow & & \\ G/P & & \end{array}$$

where $\mathcal{G}$ is equipped with a Cartan connection ω – viz. a suitably equivariant Lie(G)-valued 1-form, cf. Maurer-Cartan form on G.

E.g. Conformal geometry, projective DG, CR geometry, $\cdots$
For **conformal DG**: $G = SO_o(p+1, q+1)$, and P subgroup stabilising a ray in $\mathbb{R}^{p+q+2}$.

Tractor bundles

If we have a representation $\mathbb{V}$ of the group G then we have an associated **vector bundle** $\mathcal{G} \times_P \mathbb{V}$ with a **linear connection** ∇. This is the associated **tractor connection**. In fact for $(\mathcal{G}, \omega)$ modelled on (G, P), with G semi-simple, P parabolic:

Theorem (Čap+G.)

Cartan bundle $\mathcal{G}$ + connection $\omega \Leftrightarrow$ Tractor bundle and tractor connection.

Then:

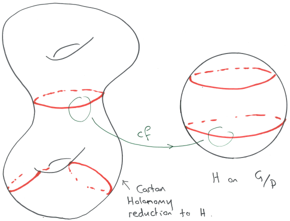
parallel tractors lead to curved analogues of orbit decompositions.

The point is that, even on the model, we can think of the orbit decomposition as arising from a parallel tractor of some type. Then the corresponding parallel tractor leads to a corresponding stratification of the curved manifold.

Theorem (**Curved orbit decomposition** - Čap,G., Hammerl)

Suppose $(\mathcal{G}, \omega) \to M$ is a Cartan geometry (modelled on $G \to G/P$) endowed with a parallel tractor field h giving a Cartan holonomy reduction with **holonomy group** *H. Then:*
*(1) M is canonically stratified $M = \bigcup_{i \in H \backslash G/P} M_i$ in a way locally diffeomorphic to the the H-***orbit decomposition** *of G/P; and*
(2) there $\exists$ a **Cartan geometry on** *M_i of the same type as the model.*

Thus there is a **general way to define a curved analogue of an orbit decomposition of a homogeneous space**.



Compactification Programme

Given some non-compact geometry of interest (e.g. pseudo-Riemannian):

Part 1 (homogeneous): Identify a homogeneous model $X_i = H/K$ of the geometry as an open $H < G$ orbit M in a compact homogeneous space $X = G/P$. (E.g. G semi-simple and P parabolic.) Then the topological closure $\overline{X}_i \subset X$ is a compactification of X_i.

Part 2 (curved I): Given a compact Cartan geometry $(\mathcal{G}, \omega) \to M$ modelled on $G \to G/P$, with a Cartan holonomy reduction with **holonomy group** H and an open curved orbit M_i (with same Cartan geometry type as X_i), then $\overline{M_i}$ is its compactification.

The Cartan/tractor machinery relates geometries of M_i & ∂M_i etc

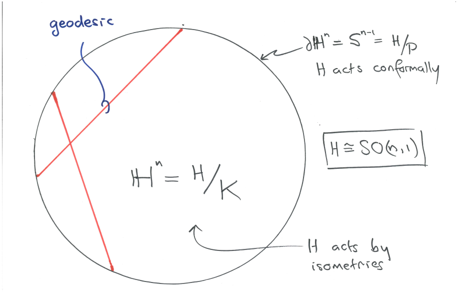
Part 3 (curved II): Typically the geometry on M_i has restrictions on e.g. Einstein or symmetries, . . . (as a **normal solution of a BGG equation holds on** M). In some cases we can drop restrictions yet still exploit the Cartan/tractor machinery.

Part IV, Projective compactification of spacetimes (and applications)

Warning: Dimension n now (not $d = n + 1$)

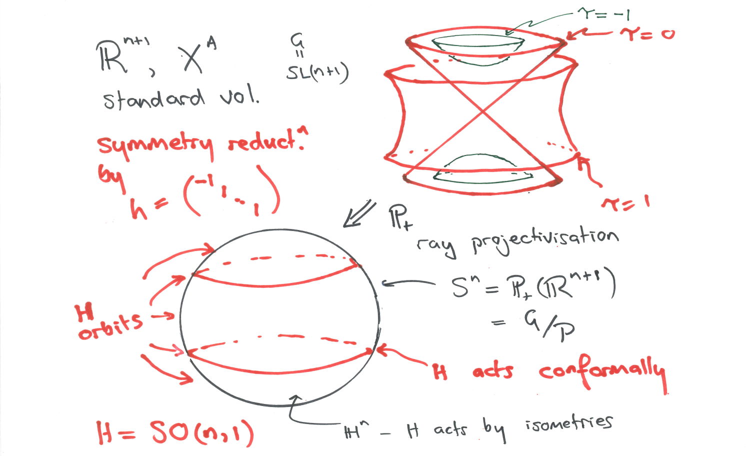
Refs, e.g.:
Čap; G-. Projective compactifications and Einstein metrics. J. Reine Angew. Math. (2016)
Čap, G-. Projective compactness and conformal boundaries. Math. Ann. (2016)
Flood, G-., Metrics in projective differential geometry: the geometry of solutions to the metrizability equation. J. Geom. Anal. (2019)

Projective compactification of hyperbolic space



The Klein ball is a **compactification** of $\mathbb{H}^n$ linked to projective geometry.

$H = SO(n,1)$ orbits on the sphere

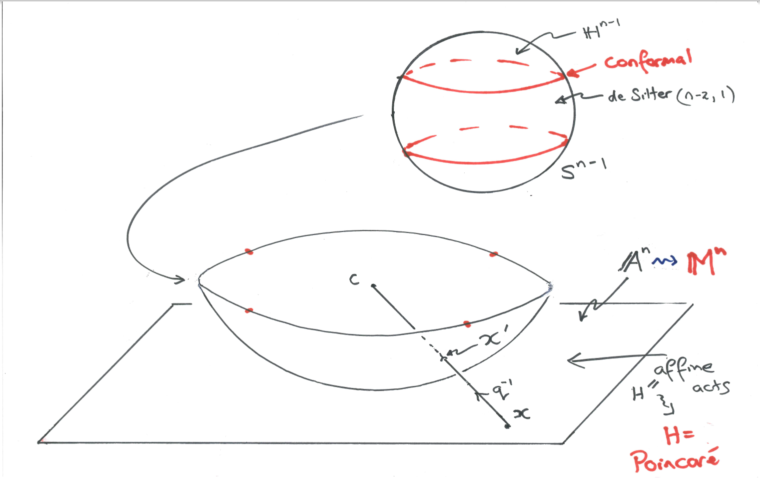


$S^n = \mathbb{P}_+(\mathbb{R}^{n+1} \setminus \{0\})$ is model of flat projective geometry.

Symmetry reduction by h (plus time↑): $\Rightarrow$ North polar cap is projective compactification of $\mathbb{H}^n$; $\tau = 0$ projective ∞ with conformal str. – also for: equatorial region which is compact'tion of **de Sitter** space

NB: Embeddings relate the orbits – also encoded in $H \hookrightarrow G$.

Projective compactification of $\mathbb{A}^n$ (and $\mathbb{M}^n$)



NB: Many geometries in one picture. How can we link all and understand how one "degenerates" into another?

Affine $= H \hookrightarrow G = SL(n+1)$ as isotropy of $I \in (\mathbb{R}^{n+1})^*$.

(*Poincare* $= H$ if also fix h^{AB} :=diag$(-1, 1, \cdots, 1, 0)$, $h^{AB} I_B = 0$.)

Projective structure

Given an affine connection ∇ and a one-form Υ on some manifold, write

$$\boxed{\widehat{\nabla} = \nabla + \Upsilon}$$

for the **projectively modified** connection defined by

$$\widehat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi, \qquad \xi, \eta \in \mathfrak{X}(M).$$

Two connections are related in this way if and only if they have the **same geodesics up to parameterisation**.

Defn: $(M, \mathbf{p})$ a **projective manifold** means $\mathbf{p} = [\nabla]$ is an equivalence class of projectively related (torsion free) affine connections.

Many equations have good projective properties, but not such good conformal properties, e.g. the geodesic equation, the Killing equation and its generalisations, the equations controlling deformations of pseudo-Riemannian geometry.

Curving: Projective DG

On a general $(M, \boldsymbol{p})$ there is no distinguished ∇ on TM. But there is on the **tractor bundle** $\mathcal{T}$ which extends TM:

$$0 \to \mathcal{E}(-1) \xrightarrow{X^A} \mathcal{T}^A \xrightarrow{Z_A^a} TM(-1) \to 0,$$

given by

$$\nabla_a^{\mathcal{T}} \begin{pmatrix} \nu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \nu^b + \rho \delta_a^b \\ \nabla_a \rho - P_{ab} \nu^b \end{pmatrix}. \quad \leftarrow \quad \textbf{standard tractor connection}$$

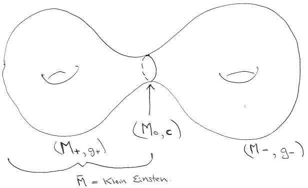
Here $(\Lambda^n TM)^2 = \mathcal{E}(2n+2)$ and $\mathcal{E}(w)$ are roots, P_{ab} =projective Schouten ($\sim$ Ricci).

Projective geometry with SO(p, q) holonomy

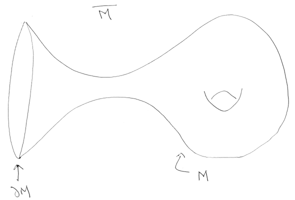
Theorem (Cap,G.,Hammerl)

*h **tractor metric** sig. (p, q) and **parallel** on (M, p) implies*

- *If $q = 0$ then $(M, \mathrm{p}, h) \Leftrightarrow (M, g)$ Einstein with positive scalar curvature.*
- *If $p, q \neq 0$ then M is stratified $M = M_+ \cup M_0 \cup M_-$ according to strict sign of $\tau = h(X, X)$.*
- *If $M_0 \neq \emptyset$ then it is a smooth embedded separating hypersurface with a **conformal structure** c of signature $(p - 1, q - 1)$.*
- *On the open submanifolds $M_\pm$, h induces metrics $g_\pm$ which are positive/negative Einstein of signature $(p - 1, q)$/ resp. $(p, q - 1)$. (Complete if M closed.)*

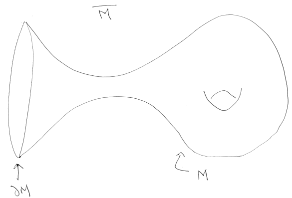


A notion of projective compactification?



Q: In general, is there a good general notion of compactification based on projective geometry?
E.g. Given a manifold with boundary, and such that the interior is equipped with a complete pseudo-Riemannian metric what should it mean to to say that $\overline{M}$ is a projective compactification of (M, g)?
Need: (1) Projective class $[\nabla^g]$ of Levi-Civita ∇^g extends smoothly to ∂M;
(2) Some uniformity in how the metric and Levi-Civita connection degenerate asymptotically [— as in conformal compactification].
(3) Analytic handle on the infinity analogous to conformal compactification, contrast: Eardley-Sachs J. Math. Phys. 1973.

Projective compactification



Defn: A torsion-free affine connection ∇ on M^{n+1} is *projectively compact* of order $\alpha \in \mathbb{R}_+$ if $\exists$ a defining function ρ for ∂M s.t.

$$\overline{\nabla} = \nabla + \frac{d\rho}{\alpha\rho}$$

extends smoothly as an affine connection to $\overline{M}$.

Proposition

If ∇ preserves a volume density vol on M then $\boxed{vol = \rho^{-\frac{n+2}{\alpha}}\overline{vol}}$ *where $\overline{vol}$ a volume density on $\overline{M}$.*

Idea of proof:

$\nabla vol = 0 \Leftrightarrow \overline{\nabla}\ \overline{vol} = 0$ therefore $\overline{vol}$ extends to $\overline{M}$. $\square$

Completeness and metrics

Theorem (Result related to completeness)

If $\alpha \leq 2$ then boundary ∂M at ∞ according to geodesics for ∇.

Defn. Metric g projectively compact order $\alpha \overset{def}{\Leftrightarrow}$ Levi-Civita ∇^g projectively compact order α.

Theorem

Let $\alpha \in (0, 2]$, s.t. $\frac{2}{\alpha} \in \mathbb{Z}$, and a metric g on M. Suppose $\exists$ a defining function ρ for ∂M s.t.

$$h := \rho^{2/\alpha} g - C \frac{d\rho \odot d\rho}{\rho^{2/\alpha}}, \qquad C \neq 0 \quad \text{constant} \tag{20}$$

extends smoothly to the ∂M with $h|_{\partial M}$ metric on $T\partial M$. **Then g is projectively compact of order α. Converse**: *g* **projectively compact of order $\alpha = 2$ then we have** *(20).*

Idea of proof $\Rightarrow$:.

Use the Koszul formula for Levi-Civita ∇^g, then show directly that $\overline{\nabla} := \nabla^g + d\rho/(\alpha\rho)$ extends smoothly to $\overline{M}$. $\square$

Re-expressing above

$$g = \frac{h}{\rho^{2/\alpha}} + C \cdot \frac{(d\rho)^2}{\rho^{4/\alpha}}$$

NB: • For $\alpha = 2$ condition (20) is independent of defining function ρ – thus:

Theorem

∂M is equipped with a canonical conformal structure.

• For $\alpha < 2$ can absorb constant C into ρ, but ρ then determined up $+O(\rho^2)$ $\therefore$ get metric on ∂M.

• For $\alpha = 1$ and $C = 1$, g has appeared in literature (R. Melrose) as Euclidean-like **"scattering metric"**.

• $\alpha = 2$ in scattering work of Vasy; noted by Fefferman-Graham as **proj. cpct** & linked to their "Ambient Metric".

The converse

$$g = \frac{h}{\rho} + C \cdot \frac{(d\rho)^2}{\rho^2}. \Leftarrow g \text{ projectively compact } \alpha = 2$$

Idea. Assume g projectively compact $\alpha = 2$.

Given a defining function ρ for ∂M and

$$\overset{\rho}{\nabla} = \nabla + \frac{d\rho}{2\rho}$$

we say a vector field μ is a *strict geodetic transversal* if $\overset{\rho}{\nabla}_\mu \mu = 0$ and $d\rho(\mu) = 1$ in nghd of ∂M. Key step: We can find such μ, ρ and $\boxed{\rho^2 g(\mu,\mu) \text{ is constant along int. curves of } \mu}$

Choose product coordinates via $t = \rho$, and x^i on ∂M. Then $t^2 g_{tt} = C(x)$. Further $C(x)$ non-zero on open dense set of ∂M. So $\partial_t(t^2 g_{tt}) = 0$ etc. Via link to Levi-Civita Christoffel symbols and projective compactness we show, where $C(x)$ not zero it is constant C, then tg_{it} and tg_{ij}, smooth to ∂M. This plus volume growth implies result.

□

Theorem

If ρ a defining function for ∂M and $g = \frac{h}{\rho} + C \cdot \frac{(d\rho)^2}{\rho^2}$ then, the curvature of g satisfies:
(i) The scalar curvature R admits a smooth extension to the boundary, with boundary value the constant $\frac{-n(n+1)}{4C}$.
(ii) The tensor field $R_{ab} + \frac{n}{4C} g_{ab}$ smoothly extends to ∂M.
(iii) Up to terms which admit a smooth extension to the boundary, the curvature of g_{ab} is given by

$$R_{ab}{}^{c}{}_{d} = -\frac{1}{2\rho^2}\delta^c_{[a}\rho_{b]}\rho_d - \frac{1}{2C\rho}\delta^c_{[a}h_{b]d}.$$

Key aspect of proof:

For *any* $\alpha = 2$ proj. compact ∇ the section $2\rho P_{ab} + \frac{1}{2\rho}\rho_a\rho_b$ extends smoothly to the boundary with value there $\overline{\nabla}_a\rho_b$ – and this a representative of the projective second fundamental form.
But it can be shown that $-\frac{1}{2C}\rho g_{ab} + \frac{1}{2\rho}\rho_a\rho_b$ has the same boundary limit (uses Koszul formula with asympt. form of g). □

Scalar curvature

For a metric with scalar curvature bounded away from zero we need only assume the projective structure extends:

Theorem

Let g be a metric on M whose projective structure smoothly extends to $\overline{M}$, but s.t. ∇^g does not admit a smooth extension to any neighborhood of a boundary point. Then

- *the scalar curvature S of g extends smoothly to ∂M.*

Furthermore for $x \in \partial M$, the following are equivalent

1. $S(x) \neq 0$
2. *g is projectively compact of order $\alpha = 2$ around x.*
3. *The boundary value of S is a non–zero constant locally around x, and g admits an asymptotic form*
$$g = C\frac{d\rho^2}{\rho^2} + \frac{h}{\rho}$$
with a constant C (related to S).

When are affine geodesics the geodesics of a metric? That is:

Q: What does it mean for there to be a Levi-Civita connection in $\boldsymbol{p}$?

An analytic answer ($n \geq 2$):

Theorem (Mikes, Sinjukov)

A special torsion-free ∇ is projectively equivalent to a Levi-Civita connection ∇^g if and only if there is a non-degenerate solution σ^{bc} to the equation

$$\text{trace-free}\left(\nabla_a \sigma^{bc}\right) = 0, \qquad \sigma \in \Gamma(S^2TM(-2)). \tag{21}$$

- This equation is **projectively invariant** – even if σ is not non-degenerate.
- In a metric projective compactification $\overline{M} = M \cup \partial M$ the boundary $\partial M = \mathcal{D}(\sigma)$ where $\mathcal{D}(\sigma)$ is the **degeneracy locus** of a solution.
- $\tau := \det(\sigma) \in \Gamma\mathcal{E}(2)$ is zero along the degeneracy locus.

The equivalent tractor equation

Theorem (Eastwood-Matveev)

The solutions to the metrisability equation trace-free $\left(\nabla_a \sigma^{bc}\right)$ *are in one-to-one correspondence with solutions of the following projectively invariant system on* $S^2\mathcal{T}$*:*

$$\tilde{\nabla}_a \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} = \nabla_a^{\mathcal{T}} \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} + \frac{1}{n} \begin{pmatrix} 0 \\ W_{ac}{}^b{}_d \sigma^{cd} \\ -2Y_{abc}\sigma^{bc} \end{pmatrix} = 0. \quad (22)$$

Here $Y_{abc} := \nabla_a P_{bc} - \nabla_b P_{ac}$ is the *projective Cotton tensor*, and $W_{ac}{}^b{}_d$ is the *projective Weyl tensor*.
And solutions of (22) are in the image of the BGG splitting operator

$$L(\sigma) = \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \sigma^{bc} & \mu^b \\ \mu^b & \rho \end{pmatrix} \in S^2\mathcal{T}.$$

where $\mu^b = -\frac{1}{n+2}\nabla_i \sigma^{ib}$ and $\rho = \frac{1}{(n+1)(n+2)}(\nabla_i \nabla_j + (n+2)\mathrm{P}_{ij})\sigma^{ij}$.

Scalar curvature extends

$L(\sigma)$ a section of $S^2\mathcal{T}$, $\therefore$ a bundle metric on $\mathcal{T}^*$. Thus $S := \det(L(\sigma))$ is well-defined and **generalises scalar curv.**:

On the interior M, $\sigma^{ab} = \tau^{-1} g^{ab}$ and in the scale g

$$L(\tau^{-1} g^{ab}) = \begin{pmatrix} \tau^{-1} g^{bc} \\ 0 \\ \frac{1}{n+1} \tau^{-1} g^{ij} P_{ij} \end{pmatrix}$$

Thus, on M, $S = \det(L(\sigma))$ is the scalar curvature (up to const. $\neq 0$). But $L(\tau^{-1} g^{ab})$ preserved by $\tilde{\nabla}$ hence extends smoothly to ∂M, thus so does S and $\tau^{-1} g^{ab}$.

NB: $0 \neq S = \det(L(\sigma)) \Leftrightarrow L(\sigma)$ is non-degenerate.

The **Proof** of the theorem $S(x) \neq 0 \Leftrightarrow g$ projectively compact of order 2 : follows by analysing $L(\sigma)$ and $L(\sigma)^{-1}$.

Projective "almost pseudo-Riemannian"

The notion of conformal almost pseudo-Riemannian has a projective analogue.
Namely: Consider $(M, \boldsymbol{p})$ equipped with a solution $\zeta \in \Gamma(S^2 TM \otimes \mathcal{E}(-2))$ of the the metrisability equation

$$\text{Trace-Free}(\nabla_a \zeta^{bc}) = 0$$

such that either

1. $L(\zeta)$ is everywhere non-degenerate – i.e. the generalised scalar curvature is nowhere zero; or
2. $L(\zeta)$ is everywhere of co-rank 1 – then the generalised scalar curvature is zero **and** (Thm) ζ **has rank** n **almost everywhere**. $< --$ This would be impossible if the co-rank was > 1.

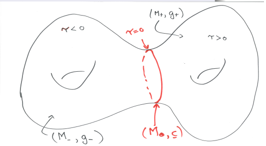
Then we have the following results:

Metrisability solutions with non-vanishing scalar curvature

Problem 1:. Suppose $(M, \boldsymbol{p})$ with ζ^{bc} s.t. Trace-Free$(\nabla_a \zeta^{bc}) = 0$ & det $L(\zeta)$ nowhere zero. What is $\mathcal{D}(\zeta)$? Geometry on it? Answer:

Theorem (Flood+G.)

Suppose $L(\zeta)$ of sig. (p, q). Either $\mathcal{D}(\zeta) = \emptyset$ or $\mathcal{D}(\zeta)$ is a smoothy embedded separating hypersurface M_0. Then:
(i) $M = M_+ \cup M_0 \cup M_-$ where ζ has signature $(p, q-1)$, $(p-1, q)$, and $(p-1, q-1, 1)$ on M_+, M_-, and M_0, respectively.
(ii) $\mathcal{D}(\zeta) = M_0$ has a conformal structure.
(iii) On $M_\pm$, ζ induces a pseudo-Riemannian metric $g_\pm$ of the same signature as ζ, where $g_\pm^{ab} = \mathrm{sgn}(\tau)\tau\zeta^{ab}|_{M_\pm}$; here $\tau = \det(\zeta)$.
(iv) If M is closed, M_+ and M_- are projectively compact of order 2, with boundary M_0. (And $\therefore$ asympt Einstein.)



Proof: Again analyse $L(\zeta)^{-1}$.

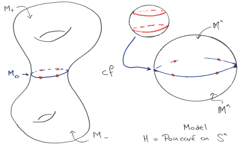
Problem 2:. linked to $\mathrm{Sc}^g = 0$ metrics we consider ME solutions ζ s.t. $\mathrm{rank}(L(\zeta)) = n$. Then what is $\mathcal{D}(\zeta)$? Geometry on it? Answer:

Theorem (Flood+G)

Let $(M, \boldsymbol{p})$ be an orientable, connected, and equipped with a ME solution ζ^{ab} s.t. $\mathrm{signat}(L(\zeta)) = (p, q, 1)$, $p + q = n$. Then
(i) $\mathcal{D}(\zeta) = \emptyset$ or $M_0 := \mathcal{D}(\zeta)$ is smoothly embedded hypersurface. If M_0 orientable then $M = M_+ \cup M_0 \cup M_-$.
(ii) $(M_\pm, g_\pm)$ are each scalar flat, pseudo-Riemannian manifolds with metric g_{ab} of signature (p, q), where $g_\pm^{ab} = \mathrm{sgn}(\tau)\tau\zeta^{ab}|_{M_\pm}$, where $\tau := \det(\zeta^{ab})$. If M is closed then $M \setminus M_\pm$ are projectively compact of order 1 with boundary M_0.
(iii) $\Sigma := M_0$ inherits a projective structure $\hat{\boldsymbol{p}}$ and a solution $\hat{\zeta}^{ab}$ of the metrisability equation with $\mathcal{D}(\hat{\zeta})$ a smoothly embedded separating hypersurface with a conformal structure, and off this we have order two projectively compact metrics $\hat{g}_\pm$.

Idea of the proof

So the Theorem says we have
a curved version of the model:



Proof: Part (i) $M_0 := \mathcal{D}(\zeta)$ is smoothly embedded hypersurface:
There is a parallel tractor volume form: $\epsilon_{A_0 A_1 \cdots A_n}$ s.t. $\nabla \epsilon = 0$.
Using this, form the (tractor) adjugate

$$\mathcal{H}_{A_0 B_0} = (-1)^q \epsilon_{A_0 A_1 \cdots A_n} \epsilon_{B_0 B_1 \cdots B_n} h^{A_1 B_1} \cdots h^{A_n B_n}.$$

Then H symmetric rank n implies

$$\mathcal{H}_{AB} = I_A I_B \quad \text{where} \quad h^{AB} I_B = 0, \quad \& \quad I_B \quad \text{nowhere } 0.$$

Then

$$I_B = \begin{pmatrix} \sigma \\ \mu_b \end{pmatrix}, \quad \mathcal{D}(\zeta) = \mathcal{Z}(\sigma) \text{ and } \zeta \text{ solves ME } \Rightarrow \quad I_B = \begin{pmatrix} \sigma \\ \nabla_b \sigma \end{pmatrix}$$

Idea of the proof ctd

Part (ii): the metrics are evident, as is the compactification because σ is both the *scale* determining these (where $\sigma \neq 0$) and it is also a weight 1 defining density for M_0 – using a result of G+Čap.

Part (iii) There are several parts:

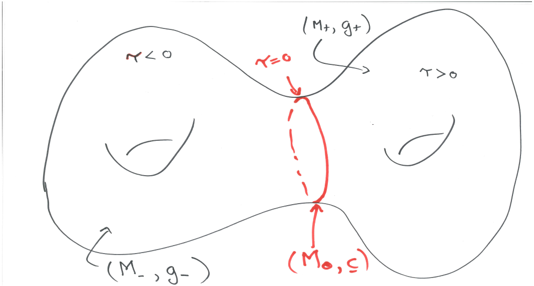
• We show prolonged system for ζ solves ME implies $\nabla I = f \cdot I$ along M_0. This shows M_0 is totally geodesic in $(M, \boldsymbol{p})$. Thus we have a projective structure $(M_0, \hat{p})$.

• Then ζ restricts along to $\hat{\zeta}$ **a solution to ME along** M_0.

• Can argue as for Theorem 41 that $\mathcal{D}(\hat{\zeta})$ is a smooth hypersurface in M_0. Then results follow *almost* as in that Theorem. □

Boundary calculus and scattering

In the setting of a Problem 2 solution, or equivalently an order 2 projectively compact (M, g)



there is again a boundary calculus and sl(2) surrounding the scattering Laplacian – Thesis of Sam Porath (U. Auckland), in progress.

Jack Borthwick, in his PhD Thesis (Brest), has extended this to Proca type equations/wave equations on differential forms.

Thank you for Listening!

THE END

$$\widetilde{g}_{ab} = \Omega^{-2} \mathrm{g}_{ab}, \quad \widetilde{T}_{ab} = \Omega^{q} \mathrm{T}_{ab}, \quad q \in \{0, 1, 2\}, \tag{23}$$

where T_{ab} is the (regular everywhere) unphysical stress-energy tensor. Then,

$$(q-2)\, n^b \mathrm{T}_{ab} + n_a \mathrm{T} = 0 \quad \text{on} \quad \Sigma, \tag{24}$$

$$\nabla_n \left[(q-2)\, \mathrm{T}_{ab} n^b + n_a \mathrm{T} \right] - \nabla_b \mathrm{T}_a{}^b = 0 \quad \text{on} \quad \Sigma, \tag{25}$$

where

$$n_a n^a = -1 + \frac{2\mathrm{H}}{\sqrt{3\Lambda}} \Omega + \mathcal{O}\left(\Omega^2\right), \tag{26}$$

H is the mean curvature of Σ, and $\mathrm{T} := \mathrm{g}^{cd} \mathrm{T}_{cd}$.