Post-Newtonian General Relativity and Gravitational Waves. Part IV: Effective One-Body Formalism

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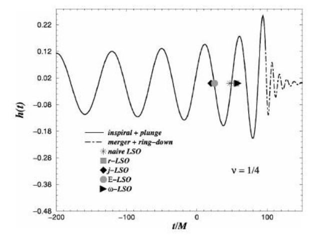
Effective One-Body Formalism

- Effective one-body (EOB) formalism provides accurate templates needed for detection of gravitational-wave signals (and estimation of their parameters) of coalescing black-hole binaries.

  EOB templates correspond to the full coalescence process of BH/BH systems from early inspiral to ringdown.
- EOB formalism is based on approximate results and it allows to model analytically motion and radiation of BH/BH system from its adiabatic inspiral, through merger, up to vibrations of the resultant Kerr BH.
- After incorporating tidal interactions EOB formalism also describes BH/NS and NS/NS systems.

EOB formalism was initiated in 1999–2000 by T.Damour and A.Buonanno at the 2PN level and is being developed since then by them and their collaborators.

T.Damour, PJ, G.Schäfer, 2000: incorporating 3PN-level orbital dynamics;
T.Damour, PJ, G.Schäfer, 2008: incorporating next-to-leading order spin-orbit corrections;
T.Damour, PJ, G.Schäfer, 2015: incorporating 4PN-level orbital dynamics.



(A.Buonanno and T.Damour, PRD, 2000)

Main Idea and Structure of EOB Approach

- Waveforms computed numerically and by means of the PN approximation of high enough order agree very well in the region, where the objects are sufficiently far away.

  Gravitational waves emitted in the last stage of the BH/BH evolution are accurately describable as a superposition of several quasi-normal modes of the Kerr BH.

- Main idea of EOB approach: extend the domain of validity of PN and BH perturbation theories up to merger and define EOB waveform as:

$$h^{\mathrm{EOB}}(t) = \theta(t_m - t)\, h^{\mathrm{ins+plunge}}(t) + \theta(t - t_m)\, h^{\mathrm{ringdown}}(t),$$

  $\theta(t)$ denotes Heaviside's step function, t_m is the time at which the two waveforms $h^{\mathrm{ins+plunge}}$ and h^{ringdown} are matched.

- Ringdown waveform $h^{\mathrm{ringdown}}(t)$ is computed from BH perturbation theory. Computation of inspiral + plunge waveform $h^{\mathrm{ins+plunge}}(t)$ requires usage of resummation techniques, which include translation of real two-body problem into effective one and usage of Padé approximants.

  The EOB approach comprises three ingredients:

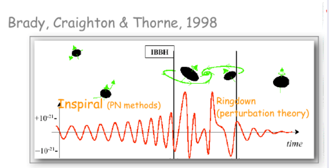
  — PN conservative Hamiltonian $\longrightarrow$ EOB-improved Hamiltonian;

  — PN gravitational-wave luminosities $\longrightarrow$ EOB radiation-reaction force;

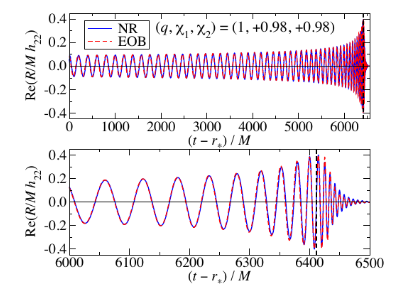
  — a description of the GW waveform emitted by a coalescing binary system.

Why Does It Work?

The merging phase could be very complicated...



but it is not...
(templates used for GW150914)



Real Two-Body Problem vs Effective One-Body Problem

$$M := m_1 + m_2, \quad \mu := \frac{m_1 m_2}{m_1 + m_2}, \quad \nu := \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}, \quad 0 \leq \nu \leq \frac{1}{4}$$

At the Newtonian level the two-body problem can be reduced to motion of a *test particle* of mass μ orbiting around an *external mass* M.
The EOB approach is a general relativistic generalization of this fact.

Real two-body problem:
two black holes of masses m_1, m_2 and spins $\mathbf{S}_1$, $\mathbf{S}_2$
orbiting around each other

$\updownarrow$

Effective one-body problem:
one test particle (with additional nongeodesic corrections)
of mass μ and spin $\mathbf{S}^*$
moving in some background metric $g_{\alpha\beta}^{\text{effective}}$

- The effective metric $g_{\alpha\beta}^{\text{effective}}$ is a ν-deformed Kerr metric of mass M and spin $\mathbf{S}_{\text{Kerr}} := \mathbf{S}_1 + \mathbf{S}_2$.
- The spin of the effective particle reads

$$\mathbf{S}^* := \frac{m_2}{m_1}\mathbf{S}_1 + \frac{m_1}{m_2}\mathbf{S}_2 + (\text{spin-orbit terms}).$$

The Mapping Rules Between the Two Problems (Motivated by Quantum Considerations)

- The adiabatic invariants (the action variables) $I_i = \oint p_i \, \mathrm{d}q_i$ are identified in the two problems.
- The energies are mapped through a function f:

$$\mathcal{E}_{\text{effective}} = f(\mathcal{E}_{\text{real}}),$$

f is determined in the process of matching.

One looks for a metric $g_{\alpha\beta}^{\text{effective}}$ such that the energies of the bound states of a particle moving in $g_{\alpha\beta}^{\text{effective}}$ are in one-to-one correspondence with the energies of the two-body bound states:

$$\mathcal{E}_{\text{effective}}(I_i) = f\left(\mathcal{E}_{\text{real}}(I_i)\right).$$

The identification of the action variables guarantees that the two problems are mapped by a canonical transformation.

Conservative 4PN-Accurate Hamiltonian Describing Relative Motion

Conservative 4PN-accurate ADM orbital Hamiltonian $H_{\leq 4\mathrm{PN}}[\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2]$ (we ignore spin-dependent corrections) is reduced to center-of-mass frame:

$$\mathbf{p}_1 + \mathbf{p}_2 = 0.$$

Rescaled dimensionless variables in the center-of-mass frame:

$$\mathbf{r} := \frac{c^2}{GM}(\mathbf{x}_1 - \mathbf{x}_2), \qquad \mathbf{p} := \frac{\mathbf{p}_1}{\mu c} = -\frac{\mathbf{p}_2}{\mu c}, \qquad \hat{t} := \frac{c^3 t}{GM}.$$

'Non-relativistic' orbital Hamiltonian:

$$\hat{H}^{\mathrm{nr}}_{\leq 4\mathrm{PN}}[\mathbf{r}, \mathbf{p}] := \frac{H_{\leq 4\mathrm{PN}}[\mathbf{r}, \mathbf{p}] - Mc^2}{\mu c^2}$$
$$= \hat{H}_{\mathrm{N}}(\mathbf{r}, \mathbf{p}) + \hat{H}_{1\mathrm{PN}}(\mathbf{r}, \mathbf{p}) + \hat{H}_{2\mathrm{PN}}(\mathbf{r}, \mathbf{p}) + \hat{H}_{3\mathrm{PN}}(\mathbf{r}, \mathbf{p}) + \hat{H}_{4\mathrm{PN}}[\mathbf{r}, \mathbf{p}],$$

$$\hat{H}_{\mathrm{N}}(\mathbf{r}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r}, \; \ldots\, .$$

Modelling Nonspinning Binaries

The effective metric is a static and spherically symmetric ν-deformation of the Schwarzschild metric:

$$g^{\rm eff}_{\mu\nu}(X';\nu)\mathrm{d}X'^{\mu}\mathrm{d}X'^{\nu} = -A(R';\nu)\,c^2\mathrm{d}T'^2 + \frac{D(R';\nu)}{A(R';\nu)}\,\mathrm{d}R'^2 + R'^2(\mathrm{d}\Theta'^2 + \sin^2\Theta'\mathrm{d}\Phi'^2),$$

metric functions A and D are looked for in the form of PN expansions [using dimensionless radial coordinate $r' := c^2R'/(GM)$],

$$A(r';\nu) = 1 + \frac{a_0(\nu)}{r'} + \frac{a_1(\nu)}{r'^2} + \frac{a_2(\nu)}{r'^3} + \frac{a_3(\nu)}{r'^4} + \frac{a_{41}(\nu) + a_{42}(\nu)\ln r'}{r'^5} + \cdots,$$

$$D(r';\nu) = 1 + \frac{d_1(\nu)}{r'} + \frac{d_2(\nu)}{r'^2} + \frac{d_3(\nu)}{r'^3} + \frac{d_{41}(\nu) + d_{42}(\nu)\ln r'}{r'^4} + \cdots.$$

- Newtonian limit: $a_0(\nu) = -2$.
- $g^{\rm eff}_{\mu\nu}$ tends to the Schwarzschild metric when $\nu \to 0$:

$$A(r';0) = 1 - \frac{2}{r'}, \quad D(r';0) = 1 \qquad \text{RESUMMATION!}$$

Effective Hamiltonian

Effective Hamiltonian H_{eff} is derived from the equation

$$\mu^2 c^2 + g_{\text{eff}}^{\mu\nu}(X') P'_\mu P'_\nu + Q(X', P') = 0, \qquad (*)$$

Q denotes contributions which are at least quartic in momenta,

$$Q(X', P') = Q_4^{\mu_1\mu_2\mu_3\mu_4}(X')\, P'_{\mu_1} P'_{\mu_2} P'_{\mu_3} P'_{\mu_4} + Q_6^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}(X')\, P'_{\mu_1} P'_{\mu_2} P'_{\mu_3} P'_{\mu_4} P'_{\mu_5} P'_{\mu_6} + \cdots .$$

One can reduce the P'-dependence of Q to a dependence on the sole $P'_r = \mathbf{n}' \cdot \mathbf{P}'$, then the equation (*) is quadratic in the time component P'_0 and $H_{\text{eff}} = -P'_0$:

$$\hat{H}_{\text{eff}}(\mathbf{r}', \mathbf{p}') = \frac{H_{\text{eff}}(\mathbf{r}', \mathbf{p}')}{\mu c^2} = \sqrt{A(r')\left(1 + \mathbf{p}'^2 + \left(\frac{A(r')}{D(r')} - 1\right)(\mathbf{n}' \cdot \mathbf{p}')^2 + \hat{Q}(\mathbf{r}', \mathbf{p}')\right)},$$

where $\mathbf{r}' := c^2\mathbf{R}/(GM)$, $\mathbf{p}' := \mathbf{P}/(\mu c)$, and $\hat{Q} := Q/(\mu^2 c^2)$.
At the 4PN accuracy $\hat{Q}$ is of the form

$$\hat{Q}(\mathbf{r}', \mathbf{p}') = \left(\frac{q_3(\nu)}{r'^2} + \frac{q_{41,4}(\nu) + q_{42,4}(\nu)\ln r'}{r'^3}\right)(\mathbf{n}' \cdot \mathbf{p}')^4 + \frac{q_{61,6}(\nu) + q_{62,6}(\nu)\ln r'}{r'^2}(\mathbf{n}' \cdot \mathbf{p}')^6 + \mathcal{O}\big((\mathbf{n}' \cdot \mathbf{p}')^8\big).$$

Map Between the Real Energy Levels and the Effective Ones

$$H_{\text{eff}} = \mu c^2 + H^{\text{nr}}\left(1 + \alpha_1 \frac{H^{\text{nr}}}{\mu c^2} + \alpha_2\left(\frac{H^{\text{nr}}}{\mu c^2}\right)^2 + \alpha_3\left(\frac{H^{\text{nr}}}{\mu c^2}\right)^3 + \alpha_4\left(\frac{H^{\text{nr}}}{\mu c^2}\right)^4 + \cdots\right),$$

for the rescaled energies it reads

$$\hat{H}_{\text{eff}}(\mathbf{r}', \mathbf{p}') = 1 + H_{\text{red}}(\mathbf{r}, \mathbf{p})\Big(1 + \alpha_1 H_{\text{red}}(\mathbf{r}, \mathbf{p}) + \alpha_2 (H_{\text{red}}(\mathbf{r}, \mathbf{p}))^2$$

$$+ \alpha_3 (H_{\text{red}}(\mathbf{r}, \mathbf{p}))^3 + \alpha_4 (H_{\text{red}}(\mathbf{r}, \mathbf{p}))^4 + \cdots\Big).$$

Split of the Real Hamiltonian

- The 4PN-accurate Hamiltonian can be decomposed in local- and nonlocal-in-time parts:

$$\hat{H}^{\mathrm{nr}}[\mathbf{r},\mathbf{p}] = \hat{H}^{\mathrm{nr\,I}}_{\mathrm{real}}(\mathbf{r},\mathbf{p};\hat{s}) + \hat{H}^{\mathrm{nr\,II}}_{\mathrm{real}}[\mathbf{r},\mathbf{p};\hat{s}],$$

$$\hat{H}^{\mathrm{nr\,I}}_{\mathrm{real}}(\mathbf{r},\mathbf{p};\hat{s}) = \hat{H}^{\mathrm{local}}_{\leq\,4\mathrm{PN}}(\mathbf{r},\mathbf{p}) + F(\mathbf{r},\mathbf{p}) \ln \frac{r}{\hat{s}},$$

$$\hat{H}^{\mathrm{nr\,II}}_{\mathrm{real}}[\mathbf{r},\mathbf{p};\hat{s}] = -\frac{1}{5}\frac{G^2}{\nu c^8}\dddot{I}_{ij}(t) \times \mathrm{Pf}_{2GM\hat{s}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\tau}{|\tau|}\dddot{I}_{ij}(t+\tau),$$

  $\hat{s} := s/(GM)$, where the scale s is a UV cutoff in the tail Hamiltonian and an IR one in the near-zone Hamiltonian [s is an intermediate scale between the size of the system r_{12} and the reduced wavelength $\lambda/(2\pi)$].

- The arbitrary scale $\hat{s}$ enters both parts, though it cancels out in the total Hamiltonian.

Split of the Effective Hamiltonian

- To the split of the real Hamiltonian, there corresponds a (4PN-accurate) split of the various building blocks A, $\bar{D}$, and $\hat{Q}$ entering the effective Hamiltonian

$$\hat{H}_{\rm eff}(\mathbf{r}',\mathbf{p}') = \sqrt{A(r')\Big(1+\mathbf{p}'^2+\Big(A(r')\bar{D}(r')-1\Big)(\mathbf{n}'\cdot\mathbf{p}')^2+\hat{Q}(\mathbf{r}',\mathbf{p}')\Big)}.$$

This split looks as follows

$$A(r') = A^{\rm I}(r') + A^{\rm II}(r'),\quad \bar{D}(r') = \bar{D}^{\rm I}(r') + \bar{D}^{\rm II}(r'),\quad \hat{Q}(\mathbf{r}',\mathbf{p}') = \hat{Q}^{\rm I}(\mathbf{r}',\mathbf{p}') + \hat{Q}^{\rm II}(\mathbf{r}',\mathbf{p}'),$$

$$A^{\rm I}(r') = 1 - \frac{2}{r'} + \frac{a_2}{r'^2} + \frac{a_3}{r'^3} + \frac{a_4}{r'^4} + \frac{a^{\rm I}_{5,\rm c} + a^{\rm I}_{5,\rm ln}\ln r'}{r'^5},\quad A^{\rm II}(r') = \frac{a^{\rm II}_{5,\rm c} + a^{\rm II}_{5,\rm ln}\ln r'}{r'^5},$$

$$\bar{D}^{\rm I}(r') = 1 + \frac{\bar{d}_1}{r'} + \frac{\bar{d}_2}{r'^2} + \frac{\bar{d}_3}{r'^3} + \frac{\bar{d}^{\rm I}_{4,\rm c} + \bar{d}^{\rm I}_{4,\rm ln}\ln r'}{r'^4},\quad \bar{D}^{\rm II}(r') = \frac{\bar{d}^{\rm II}_{4,\rm c} + \bar{d}^{\rm II}_{4,\rm ln}\ln r'}{r'^4},$$

$$\hat{Q}^{\rm I}(\mathbf{r}',\mathbf{p}') = \left(\frac{q_{42}}{r'^2} + \frac{q^{\rm I}_{43,\rm c} + q^{\rm I}_{43,\rm ln}\ln r'}{r'^3}\right)(\mathbf{n}'\cdot\mathbf{p}')^4 + \frac{q^{\rm I}_{62,\rm c} + q^{\rm I}_{62,\rm ln}\ln r'}{r'^2}(\mathbf{n}'\cdot\mathbf{p}')^6,$$

$$\hat{Q}^{\rm II}(\mathbf{r}',\mathbf{p}') = \frac{q^{\rm II}_{43,\rm c} + q^{\rm II}_{43,\rm ln}\ln r'}{r'^3}(\mathbf{n}'\cdot\mathbf{p}')^4 + \frac{q^{\rm II}_{62,\rm c} + q^{\rm II}_{62,\rm ln}\ln r'}{r'^2}(\mathbf{n}'\cdot\mathbf{p}')^6 + \mathcal{O}((\mathbf{n}'\cdot\mathbf{p}')^8).$$

- After employing the I + II split of the functions A, $\bar{D}$, and $\hat{Q}$, one expands $\hat{H}_{\rm eff}$ into a PN Taylor series (i.e., with respect to $\mathbf{p}'^2 \sim 1/r' \sim 1/c^2$). One gets

$$\hat{H}_{\rm eff}(\mathbf{r}',\mathbf{p}') = \hat{H}^{\rm I}_{\rm eff}(\mathbf{r}',\mathbf{p}') + \hat{H}^{\rm II}_{\rm eff}(\mathbf{r}',\mathbf{p}') + \mathcal{O}(c^{-10}).$$

Matching of the Local Part of the Hamiltonian

- The identification of the action variables guarantees that the two problems are mapped by a *canonical transformation*, with generating function

$$\tilde{\mathfrak{g}}_{\leq 4\mathrm{PN}}(\mathbf{r}, \mathbf{p}') = \mathbf{r} \cdot \mathbf{p}' + \mathfrak{g}_{\leq 4\mathrm{PN}}(\mathbf{r}, \mathbf{p}'),$$

  so the relation between the real phase-space coordinates $(\mathbf{r}, \mathbf{p})$ and the effective phase-space coordinates $(\mathbf{r}', \mathbf{p}')$ reads

$$x'^i = x^i + \frac{\partial \mathfrak{g}_{\leq 4\mathrm{PN}}(\mathbf{r}, \mathbf{p}')}{\partial p'_i}, \quad p_i = p'_i + \frac{\partial \mathfrak{g}_{\leq 4\mathrm{PN}}(\mathbf{r}, \mathbf{p}')}{\partial x^i}.$$

- The generating function has the symbolic structure

$$\mathfrak{g}_{\leq 4\mathrm{PN}}(\mathbf{r}, \mathbf{p}') = \mathfrak{g}_{\leq 3\mathrm{PN}}(\mathbf{r}, \mathbf{p}') + (\mathbf{r} \cdot \mathbf{p}')(1 + \ln r) \times \left((\mathbf{p}'^2)^4 + \frac{1}{r}\big((\mathbf{p}'^2)^3 + \cdots\big) + \cdots + \frac{1}{r^4} \right).$$

Matching of the Nonlocal Part of the Hamiltonian (1/6)

- One can reduce nonlocal-in-time dynamics to local-in-time one by means of Delaunay (action-angle) reduction.
- The action-angle variables for Newtonian motion in the fixed plane ($\hat{a} := a/(GM)$ is semimajor axis and e is eccentricity):
  - ℓ is the mean anomaly, its conjugate $\mathcal{L} := \sqrt{\hat{a}}$,
  - $g \equiv \omega$ is the argument of the periastron, its conjugate $\mathcal{G} := \sqrt{\hat{a}(1-e^2)}$.
- The mean anomaly ℓ is an angle that increases uniformly in time at the rate of 2π radians every orbital period.
  The argument of the periastron g is the angle subtended between the direction of the ascending node and the direction of the orbit's periastron.

Matching of the Nonlocal Part of the Hamiltonian (2/6)

- The Newtonian Delaunay Hamiltonian,

$$\hat{H}_{\mathrm{N}}(\ell, g, \mathcal{L}, \mathcal{G}) = \frac{1}{2}\,\mathbf{p}^2 - \frac{1}{r} = -\frac{1}{2\mathcal{L}^2}.$$

- Equations of motion

$$\frac{\mathrm{d}\ell}{\mathrm{d}\hat{t}} = \frac{\partial \hat{H}_{\mathrm{N}}}{\partial \mathcal{L}} = \frac{1}{\mathcal{L}^3} \equiv \hat{\Omega}(\mathcal{L}), \quad \frac{\mathrm{d}g}{\mathrm{d}\hat{t}} = \frac{\partial \hat{H}_{\mathrm{N}}}{\partial \mathcal{G}} = 0,$$

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\hat{t}} = -\frac{\partial \hat{H}_{\mathrm{N}}}{\partial \ell} = 0, \quad \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\hat{t}} = -\frac{\partial \hat{H}_{\mathrm{N}}}{\partial g} = 0,$$

where the time variable $\hat{t} := t/(GM)$ and $\hat{\Omega}$ is the rescaled Newtonian orbital frequency. It satisfies the rescaled Kepler law:

$$\hat{\Omega} = \hat{a}^{3/2}.$$

Elimination of periodically varying terms

- The expression (which enters the nonlocal-in-time piece $\hat{H}^{\rm nr\,II}_{\rm real}$)
$$\mathcal{F}(t,\tau) := \dddot{I}_{ij}(t)\,\dddot{I}_{ij}(t+\tau),$$
can be rewritten as (here n_1, n_2, n_3 are positive integers)
$$\mathcal{F}(\ell,\hat{\tau}) = \sum_{n_1,n_2,\pm n_3} C^{\pm}_{n_1 n_2 n_3}\, e^{n_1} \cos(n_2\ell \pm n_3\Omega(\mathcal{L})\hat{\tau}).$$
After integarting over $\hat{\tau}$ any term containing $n_2 \neq 0$ generates $\propto \cos(n_2\ell)$ contribution to $\hat{H}^{\rm nr\,II}_{\rm real}$.
- Any term of the type $A(\mathcal{L})\cos(n\ell)$ in a first-order perturbation $\epsilon H_1(\ell,\mathcal{L})$ can be eliminated by a canonical transformation with generating function of the type $\mathfrak{g}(\mathcal{L},\ell) = B(\mathcal{L})\sin(n\ell)$:
$$\delta_{\mathfrak{g}} H_1 = \{\hat{H}_{\rm N}(\mathcal{L}),\mathfrak{g}\} = -\frac{\partial \hat{H}_{\rm N}(\mathcal{L})}{\partial \mathcal{L}}\,\frac{\partial \mathfrak{g}}{\partial \ell} = -n\,\Omega(\mathcal{L})\,B(\mathcal{L})\cos(n\ell),$$
the choice $B := A/(n\,\Omega)$ eliminates the term $A\cos(n\ell)$ in $\hat{H}^{\rm nr\,II}_{\rm real}$.

ℓ-averaged Hamiltonian

- One can simplify the 4PN Hamiltonian $\hat{H}^{\mathrm{nr\,II}}_{\mathrm{real}}$ by replacing it by its ℓ-averaged value,

$$\hat{\bar{H}}^{\mathrm{nr\,II}}_{\mathrm{real}}(\mathcal{L},\mathcal{G};\hat{s}) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\ell\, \hat{H}^{\mathrm{nr\,II}}_{\mathrm{real}}[\mathbf{r},\mathbf{p};\hat{s}] = -\frac{1}{5}\,\frac{G^2}{\nu c^8}\,\mathrm{Pf}_{2\hat{s}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\hat{\tau}}{|\hat{\tau}|}\,\bar{\mathcal{F}},$$

  where $\bar{\mathcal{F}}$ denotes the ℓ-average of $\mathcal{F}(\ell,\hat{\tau})$.

- $\hat{\bar{H}}^{\mathrm{nr\,II}}_{\mathrm{real}}(\mathcal{L},\mathcal{G};\hat{s})$ is given as an expansion in (only even) powers of e.
- One can employ the Bessel-Fourier expansion of the quadrupole moment ($\mathrm{e} = 2.718\ldots$ should be distinguished from the eccentricity e)

$$I_{ij}(\ell;e) = \sum_{p=-\infty}^{+\infty} I^p_{ij}(e)\,\mathrm{e}^{ip\ell}.$$

Explicit form of $\hat{\hat{H}}^{\mathrm{nr\,II}}_{\mathrm{real}}(\mathcal{L}, \mathcal{G}; \hat{s})$

$$\hat{H}^{\mathrm{nr\,II}}_{\mathrm{real}}(\mathcal{L}, \mathcal{G}; \hat{s}) = \frac{4}{5}\frac{G^2}{\nu c^8}\left(\frac{\Omega}{GM}\right)^6 \sum_{p=1}^{\infty} p^6 |I^p_{ij}(e)|^2 \ln\left(2p\frac{\mathrm{e}^{\gamma_\mathrm{E}}\Omega\hat{s}}{c}\right)$$

$$= \frac{\nu}{c^8\mathcal{L}^{10}}\left(\frac{64}{5}\left(2\ln 2 + \ln\left(\frac{\mathrm{e}^{\gamma_\mathrm{E}}\hat{s}}{c\mathcal{L}^3}\right)\right) + \frac{1}{5}\left(\frac{296}{3}\ln 2 + 729\ln 3 + \frac{1256}{3}\ln\left(\frac{\mathrm{e}^{\gamma_\mathrm{E}}\hat{s}}{c\mathcal{L}^3}\right)\right)e^2\right.$$

$$+\left(\frac{29966}{15}\ln 2 - \frac{13851}{20}\ln 3 + 242\ln\left(\frac{\mathrm{e}^{\gamma_\mathrm{E}}\hat{s}}{c\mathcal{L}^3}\right)\right)e^4$$

$$\left.+\left(-\frac{116722}{15}\ln 2 + \frac{419661}{320}\ln 3 + \frac{1953125}{576}\ln 5 + \frac{1526}{3}\ln\left(\frac{\mathrm{e}^{\gamma_\mathrm{E}}\hat{s}}{c\mathcal{L}^3}\right)\right)e^6 + \mathcal{O}(e^8)\right).$$

Matching of the Nonlocal Part of the Hamiltonian (6/6)

Explicit form of $\hat{H}^{\rm II}_{\rm eff}(\mathcal{L},\mathcal{G})$

$$\hat{H}^{\rm II}_{\rm eff}(\mathcal{L},\mathcal{G}) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\ell\, \hat{H}^{\rm II}_{\rm eff}[\mathbf{r}', \mathbf{p}']$$

$$= \frac{1}{2\mathcal{L}^{10}}\Bigg(a^{\rm II}_{5,\rm c} + a^{\rm II}_{5,\rm ln}\ln(\mathcal{L}^2) + \frac{1}{4}\left(20a^{\rm II}_{5,\rm c} - 9a^{\rm II}_{5,\rm ln} + 2\bar{d}^{\rm II}_{4,\rm c} + 2(10a^{\rm II}_{5,\rm ln} + \bar{d}^{\rm II}_{4,\rm ln})\ln(\mathcal{L}^2)\right)e^2$$

$$+\left(\frac{1}{8}\left(105a^{\rm II}_{5,\rm c} - \frac{319}{4}a^{\rm II}_{5,\rm ln} + 15\bar{d}^{\rm II}_{4,\rm c} - \frac{11}{2}\bar{d}^{\rm II}_{4,\rm ln} + 3q^{\rm II}_{43,\rm c}\right) + \frac{3}{8}(35a^{\rm II}_{5,\rm ln} + 5\bar{d}^{\rm II}_{4,\rm ln} + q^{\rm II}_{43,\rm ln})\ln(\mathcal{L}^2)\right)e^4$$

$$+\Bigg(\frac{1}{192}\left(5040a^{\rm II}_{5,\rm c} - 5018a^{\rm II}_{5,\rm ln} + 840\bar{d}^{\rm II}_{4,\rm c} - 533\bar{d}^{\rm II}_{4,\rm ln} + 252q^{\rm II}_{43,\rm c} - 78q^{\rm II}_{43,\rm ln} + 60q^{\rm II}_{62,\rm c}\right)$$

$$+\frac{1}{16}(420a^{\rm II}_{5,\rm ln} + 70\bar{d}^{\rm II}_{4,\rm ln} + 21q^{\rm II}_{43,\rm ln} + 5q^{\rm II}_{62,\rm ln})\ln(\mathcal{L}^2)\Bigg)e^6 + \mathcal{O}(e^8)\Bigg).$$

The matching equation

$$\hat{H}^{\rm II}_{\rm eff}(\mathcal{L},\mathcal{G}) = \hat{H}^{\rm nr\,II}_{\rm real}(\mathcal{L},\mathcal{G})$$

leads to *unique* values of the coefficient of $\hat{H}^{\rm II}_{\rm eff}(\mathcal{L},\mathcal{G})$.

The Real EOB-Improved 4PN-Accurate Hamiltonian

- Results of the 4PN-accurate matching for energy map:

$$\alpha_1 = \frac{\nu}{2}, \quad \alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 0.$$

- The energy map can be written as

$$\frac{H_{\text{eff}}}{\mu c^2} = \frac{H_{\text{real}}^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4},$$

from this the real EOB-improved 4PN-accurate Hamiltonian follows:

$$H_{\text{real}}^{\text{EOB-improved}} = M c^2 \sqrt{1 + 2\nu \left(\frac{H_{\text{eff}}}{\mu c^2} - 1 \right)}.$$

Radiation Reaction for Quasi-Circular Motions (1/2)

- We restrict to the planar dynamics described by the local-in-time real EOB-improved 3PN-accurate Hamiltonian (we omit the primes in the canonical variables) $H_{\text{real}}^{\text{EOB-improved}}(r, p_r, p_\phi)$. It gives the following conservative equations of motion:

$$\dot{r} = \frac{\partial H_{\text{real}}^{\text{EOB-improved}}(r, p_r, p_\phi)}{\partial p_r},$$

$$\dot{\phi} = \frac{\partial H_{\text{real}}^{\text{EOB-improved}}(r, p_r, p_\phi)}{\partial p_\phi},$$

$$\dot{p}_r = -\frac{\partial H_{\text{real}}^{\text{EOB-improved}}(r, p_r, p_\phi)}{\partial r},$$

$$\dot{p}_\phi = 0.$$

Radiation Reaction for Quasi-Circular Motions (2/2)

- For quasi-circular motions ($|\dot{r}| \ll r\dot{\phi}$) it is enough, too a good approximation, to add only the ϕ component of the damping force, what modifies the equation for $\dot{p}_\phi$,

$$\dot{p}_\phi = \mathcal{F}_\phi .$$

- As $\dot{p}_\phi$ is just the total angular momentum of the binary system, the above equation expresses the rate of loss of angular momentum under gravitational radiation reaction.
  In the case of quasi-circular orbits there is a simple relation between angular momentum loss $-\mathcal{F}_\phi(\dot{\phi})$ and energy loss $\mathcal{L}(\dot{\phi})$,

$$\mathcal{L}(\dot{\phi}) = -\dot{\phi}\mathcal{F}_\phi(\dot{\phi}).$$

  Finally,

$$\dot{p}_\phi = -\frac{1}{\dot{\phi}}\mathcal{L}(\dot{\phi}).$$

Padé Approximants

Padé approximant of (k, l)-type (with $k + l = n$) for series
$w(x) = c_0 + c_1\, x + \cdots + c_n\, x^n$ (with $c_0 \neq 0$):

$$\mathrm{P}_l^k(w(x)) := \frac{N_k(x)}{D_l(x)},$$

where the polynomials N_k (of degree k) and D_l (of degree l) are such that the Taylor expansion of $\mathrm{P}_l^k(w(x))$ coincides with $w(x)$ up to $\mathcal{O}(x^{n+1})$ terms:

$$\frac{N_k(x)}{D_l(x)} = c_0 + c_1\, x + \cdots + c_n\, x^n + \mathcal{O}(x^{n+1}).$$

Last (Innermost) Stable Cirular Orbit

- The reduced angular momentum of the system in the center-of-mass reference frame:

$$j := \frac{\mathcal{J}}{G m_1 m_2}.$$

- In the test-mass limit and along circular orbits:

$$j(x;\nu=0) = \frac{1}{\sqrt{x(1-3x)}}, \quad x := \frac{1}{c^2}(GM\dot{\phi})^{2/3}.$$

  the pole $x = 1/3$ corresponds to light ring (the last unstable circular orbit).

- In the test-mass limit ($\nu = 0$) LSCO is for $x = 1/6$, which is the minimum of the function $j^2(x;0)$.

  For $\nu \neq 0$ one defines the location od the LSCO as the minimum of the function $j^2(x;\nu)$.

Last (Innermost) Stable Cirular Orbit for $\nu \neq 0$

4PN-accurate PN computations give

$$j^2(x;\nu) = \frac{1}{x}\left(1 + \left(3 + \frac{1}{3}\nu\right)x + j_2(\nu)\,x^2 + j_3(\nu)\,x^3 + (j_{41}(\nu) + j_{42}(\nu)\ln x)\,x^4\right).$$

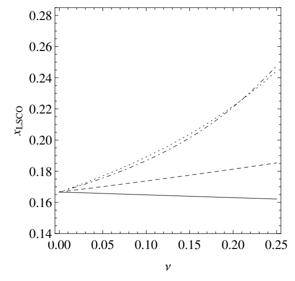
One constructs the sequence of Padé approximants of $j^2(x;\nu)$:

$$j^2_{P_1}(x;\nu) := \frac{1}{x}\,\mathrm{P}^0_1\left[1 + (3 + \frac{1}{3}\nu)\,x\right] = \frac{1}{x\left(1 - \left(3 + \frac{1}{3}\nu\right)x\right)},$$

$$j^2_{P_2}(x;\nu) := \frac{1}{x}\,\mathrm{P}^1_1\left[1 + (3 + \frac{1}{3}\nu)\,x + j_2(\nu)\,x^2\right],$$

$$j^2_{P_3}(x;\nu) := \frac{1}{x}\,\mathrm{P}^2_1\left[1 + (3 + \frac{1}{3}\nu)\,x + j_2(\nu)\,x^2 + j_3(\nu)\,x^3\right],$$

$$j^2_{P_4}(x;\nu) := \frac{1}{x}\left(\mathrm{P}^3_1\left[1 + (3 + \frac{1}{3}\nu)\,x + j_2(\nu)\,x^2 + j_3(\nu)\,x^3 + j_{41}(\nu)\,x^4\right] + j_{42}(\nu)\,x^4\ln x\right).$$



At all PN levels the test-mass result is recovered exactly:

$$\lim_{\nu\to 0} j^2_{P_n}(x;\nu) = \frac{1}{x(1-3x)} \quad \text{for } n = 1,2,3,4.$$

Padé-Improved EOB Potential $A(u;\nu)$

The EOB potential $A(u;\nu) = -g_{00}^{\text{eff}}(u;\nu)$ (with $u := 1/r'$) has the following 4PN-accurate Taylor expansion:

$$A(u;\nu) = 1 - 2u + 2\nu u^3 + a_4(\nu)u^4 + \big(a_{51}(\nu) + a_{52}(\nu)\ln u\big)u^5 + \mathcal{O}(u^6).$$

By continuity with the test-mass case $\nu \to 0$, one expects that $A(u;\nu)$ will exhibit a simple zero defining an EOB "effective horizon" that is smoothly connected, when $\nu \to 0$, to the Schwarzschild event horizon at $u = 1/2$.

Therefore it is reasonable to factor a zero of $A(u;\nu)$ by introducing the Padé-improved $A_{P_4}(u;\nu)$ defined at the 4PN level as

$$A_{P_4}(u;\nu) := \mathrm{P}_4^1\left[1 - 2u + 2\nu u^3 + a_4(\nu)u^4 + (a_{51}(\nu) + a_{52}(\nu)\ln u)u^5\right].$$

Synergy Between EOB Formalism and Numerical Relativity

Select sample of NR waveforms

$\downarrow$

Introduce EOB flexibility parameters
and calibrate them to NR waveforms

$\downarrow$

Define NR-improved EOB waveforms
(used in analysis of LIGO/Virgo data)

Flexibility Parameters in the EOB Potential $A(u;\nu)$ (for nonspinning BH/BH systems)

Instead of using the 4PN-accurate truncated Taylor expansion (maybe in Padé-improved form),

$$A^{4\mathrm{PN}}(u;\nu) = 1 - 2u + 2\nu u^3 + a_4(\nu)u^4 + (a_{51}(\nu) + a_{52}(\nu)\ln u)u^5,$$

one considers (maybe in Padé-improved form) 3-parameter class of extensions of $A^{4\mathrm{PN}}(u;\nu)$ defined by

$$A(u;\nu, b_{61}, b_{62}, b_{63}) := A^{4\mathrm{PN}}(u;\nu) + \nu(b_{61} + b_{62}\nu + b_{63}\nu^2)u^6 + a_{62}(\nu)u^6 \ln u.$$

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