# Post-Newtonian General Relativity and Gravitational Waves. Part I: Post-Newtonian (PN) Two-Body Problem and Gravitational-Wave (GW) Astronomy


Piotr Jaranowski

Faculty of Physics, University of Białystok, Poland


**School of General Relativity, Astrophysics and Cosmology**

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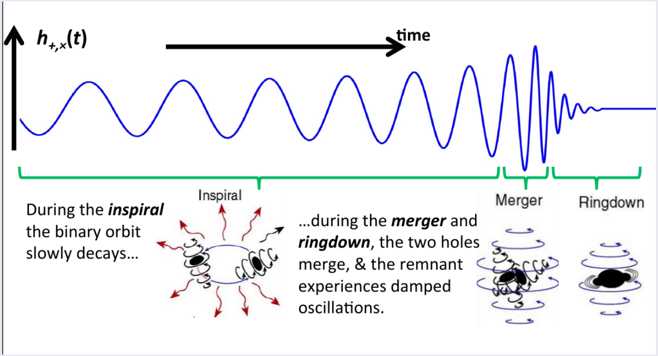
# Gravitational-Wave Signal from Coalescence of Black-Hole Binary

A laser-interferometric detector measures differential displacement along the detector's arms:

$$\boxed{\Delta L(t) = \delta L_x(t) - \delta L_y(t) = h(t)\, L_0, \quad h(t) \sim 10^{-21},}$$

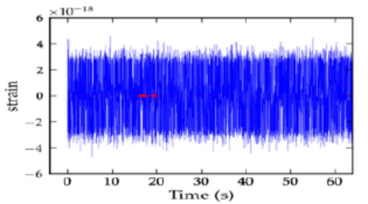
where $L_0 = 4$ km for LIGO detectors or $L_0 = 3$ km for Virgo/KAGRA detectors, $h(t)$ is the dimensionless gravitational-wave strain.




The detected waveforms match the predictions of general relativity for the inspiral and merger of a pair of stellar-mass black holes and the ringdown of the resulting single Kerr black hole.

# Very Sensitive Search Algorithms

The expected gravitational-wave signals are weak compared to instrumental noise,
therefore the detection of signals and identification of their sources requires employing very sensitive search procedures.




- Each search procedure defines a detection statistic that ranks likelihood of presence a gravitational-wave signal in the data.
  One then identifies candidate events of high enough value of the detection statistic that are detected at all operating observatories and their time of arrivals are consistent with the intersite propagation times.
- The significance of a candidate event is determined by false alarm rate (FAR): the rate at which detector noise produces events with a detection-statistic value equal to or higher than the detection-statistic value of the candidate event.
- This translates to a false alarm probability: a probability of observing one or more noise events as strong as the candidate event during the analysis time.

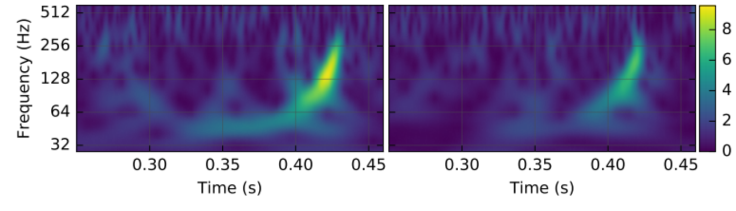
## Two Main Types of Search Algorithms

Two types of search algorithms are used in data analysis:

- searches for generic gravitational-wave transients, which operate without a specific waveform model and identify coincident excess power in a time-frequency representation of the data;
- matched-filtering searches using relativistic models of compact binary coalescence waveforms, they correlate the data with a copy of a waveform, i.e. a template, one expects to find in the data.

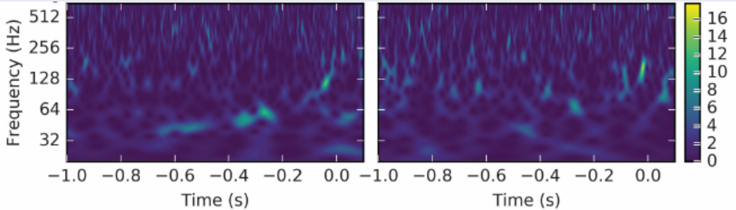
# Results of Generic Transient Searches (from Observing Run O1)

GW150914




Significance of GW150914: FAR $< 1.00 \times 10^{-7}$ yr$^{-1}$ (GstLAL pipeline).

GW151226




Significance of GW151226: FAR $< 1.00 \times 10^{-7}$ yr$^{-1}$ (GstLAL pipeline).

Normalized amplitude of a time-frequency representation of the strain data (left for LIGO Hanford, right for LIGO Livingston detector)

## Advantages and Challenges of Matched Filtering

### Advantages

- Matched filtering has better sensitivity than unmodeled searches.
- Constraining space of possible signals decreases false alarm rate.
- One can use signal-based vetoes to separate signals from transient noise.

### Challenges

It loses sensitivity if templates do not match signals:

- accurate waveform models, i.e. templates, are needed;
- parameters of template must be close enough to signal: templates' parameters have to cover space spanned by the signal's parameters densly enough.

One has to construct a discrete bank of templates parametrized by possible values of the gravitational-wave signal's parameters:

$$h(t; m_1, m_2, \mathsf{S}_1, \mathsf{S}_2, \Lambda_1, \Lambda_2, \ldots),$$

where $m_1$, $m_2$ and $\mathsf{S}_1$, $\mathsf{S}_2$ are masses and spins of binary components, $\Lambda_1$, $\Lambda_2$, ... are tidal parameters.

Construction of accurate templates requires accurate enough solution of relativistic two-body problem for each value of signal's parameter present in the bank of templates.

## Matched-Filtering Search (1/2)

- For each template $h(t)$ and for the strain data from a single detector $s(t)$, the analysis calculates the matched-filter signal-to-noise ratio

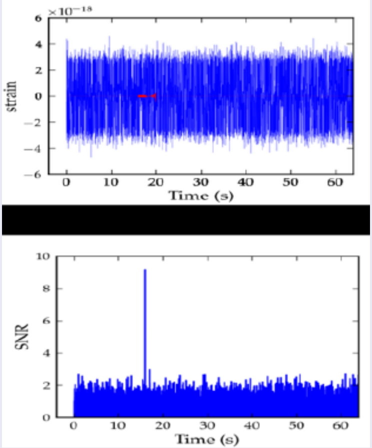
$$\rho(t_a) := \frac{|\langle s|h\rangle(t_a)|}{\sqrt{\langle h|h\rangle}},$$

where $t_a$ is time of arrival of the signal and the correlation is defined as

$$\langle s|h\rangle(t_a) = 4\int_0^\infty \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_n(f)} \mathrm{e}^{2\pi i f t_a}\mathrm{d}f,$$

$S_n(f)$ is the one-sided (average) power spectral density of the detector noise, $\tilde{s}(f)$ is the Fourier transform of $s(t)$,

$$\tilde{s}(f) := \int_{-\infty}^{\infty} s(t)\mathrm{e}^{-2\pi i f t}\mathrm{d}t.$$



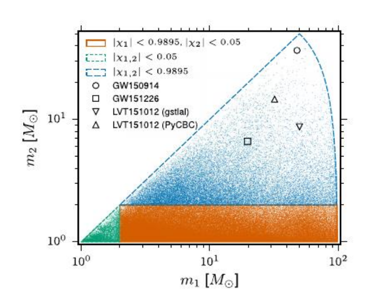

- $\rho(t_a)$ is maximized with respect to the time of arrival of the signal.

## Matched-Filtering Search (2/2)

- Each maximal value of $\rho(t_\mathrm{a})$ is reweighted by the value of a chi-squared statistic testing whether the data in several different frequency bands are consistent with the template.
- When, say, two LIGO detectors are operating:
  - — event pairs that occur within a 15-ms window and come from the same template are selected (the 15-ms window is determined by the 10-ms intersite propagation time plus 5 ms for uncertainty in arrival time of weaker signals);
  - — the quadrature sum of the reweighted $\rho(t_\mathrm{a})$ of two coincident events is calculated.

## Binary Coalescence Searches in Observing Run O1

- The waveforms depend on the masses $m_1$, $m_2$ of the binary components and their dimensionless spins $|\chi_1|$ and $|\chi_2|$ [$\chi_a := c\mathbf{S}_a/(Gm_a^2)$, $a = 1, 2$].
- The searches targeted binaries with individual masses from 1 to $99M_\odot$, total mass not greater than $100M_\odot$, and dimensionless spins up to 0.9895.
- Waveforms modeling systems with total mass less than $4M_\odot$: PN inspiral waveforms accurate to 3.5PN order.
- Waveforms modeling systems with total mass larger than $4M_\odot$: inspiral+merger+ringdown waveforms constructed by means of the effective-one-body formalism.
- Around 250 000 templates were used.




The waveform model assumes that the spins of the merging objects are aligned with the orbital angular momentum, but the resulting templates can recover systems with misaligned spins (in the parameter region of detected signals).

## Computing Power Constraints

- Construction of bank of templates requires multiple integration of

  — partial differential equations in numerical relativity,
  — ordinary differential equations in approximate analytical relativity.

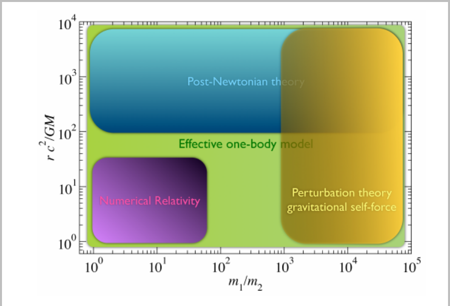
- For detection of gravitational-wave signals originated from coalescences of binaries made of spinning black holes/neutron stars with arbitrary mass ratios, due to limitations in available computing power, it will not be possible in the nearest future to construct bank of templates based purely on numerical results.

## Different Approaches for Solving Relativistic Two-Body Problem

- Numerical relativity (breakthrough in 2005).
- Approximate 'analytical' methods:
  - — post-Newtonian (PN) expansion,
  - — post-Minkowskian (PM) expansion,
  - — black-hole perturbation self-force based approach (Adam Pound's lectures).

  The adjective "analytical" means here methods that rely on solving explicit (that is analytically given) ordinary differential equations, contrary to full numerical relativity simulations that involve solving systems of partial differential equations.
- Effective-one-body formalism (EOB): combines results of PN/PM approaches, black-hole perturbation theory, and numerical relativity.

- PN expansion:
  0th order—Newtonian gravity;
  $n$PN order—corrections of order
  $$\left(\frac{v}{c}\right)^{2n} \sim \left(\frac{Gm}{rc^2}\right)^n$$
  to the Newtonian gravity.
- PM expansion: expansion in powers of $G$.
- Perturbation approach: $\dfrac{m_1}{m_2} \gg 1$.




- The $n$PM-order expansion controls all terms in the corresponding PN approximation through $(n-1)$PN order.

## Different "Flavours" of the PN/PM Expansions

- ADM Hamiltonian approach (Damour/Jaranowski/Schäfer).
- Harmonic-coordinate based direct iteration (Blanchet *et al.*).
- Effective field theory approach: advanced calculations of scattering amplitudes using generalized unitarity, double-copy construction, eikonal resummation, and advanced multiloop integration methods.
- "Tutti frutti" approach (Bini/Damour/Geralico) combines various analytical approximation methods: PN, PM, multipolar post-Minkowskian, effective field theory, gravitational self-force, effective one body, and Delaunay averaging.

## Gravitational Waves from Inspiralling Binary on Circular Orbits

The gravitational-wave strain measured by the laser-interferometric detector and induced by gravitational waves from coalescing compact binary made of nonspinning bodies in circular orbits during inspiral phase:

$$\boxed{h(t) = \frac{C}{D}\left[\dot{\phi}(t)\right]^{2/3} \sin\left[2\phi(t) + \alpha\right],}$$

where $\phi(t)$ is the orbital phase of the binary [so $\dot{\phi}(t) := \mathrm{d}\phi(t)/\mathrm{d}t$ is the angular frequency], $D$ is the luminosity distance of the binary to the Earth, $C$ and $\alpha$ are some constants.

The time evolution of the orbital phase $\phi(t)$ is computed from the balance equation:

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{L} \quad \Longrightarrow \quad \phi = \phi(t),}$$

which both sides have the following PN expansions:

$$E = \boxed{E_{\mathrm{N}}} + \frac{1}{c^2}\boxed{E_{1\mathrm{PN}}} + \frac{1}{c^4}\boxed{E_{2\mathrm{PN}}} + \frac{1}{c^6}\boxed{E_{3\mathrm{PN}}} + \frac{1}{c^8}\boxed{E_{4\mathrm{PN}}} + \mathcal{O}\big((v/c)^{10}\big),$$

$$\begin{aligned}\mathcal{L} = \boxed{\mathcal{L}_{\mathrm{N}}} &+ \frac{1}{c^2}\boxed{\mathcal{L}_{1\mathrm{PN}}} + \frac{1}{c^3}\boxed{\mathcal{L}_{1.5\mathrm{PN}}} + \frac{1}{c^4}\boxed{\mathcal{L}_{2\mathrm{PN}}} + \frac{1}{c^5}\boxed{\mathcal{L}_{2.5\mathrm{PN}}} \\ &+ \frac{1}{c^6}\boxed{\mathcal{L}_{3\mathrm{PN}}} + \frac{1}{c^7}\boxed{\mathcal{L}_{3.5\mathrm{PN}}} + \frac{1}{c^8}\boxed{\mathcal{L}_{4\mathrm{PN}}} + \frac{1}{c^9}\boxed{\mathcal{L}_{4.5\mathrm{PN}}} + \mathcal{O}\big((v/c)^{10}\big).\end{aligned}$$

# 4.5PN-Accurate Binding Energy in the Center-of-Mass Frame for Circular Orbits

- **Notation**

  Masses of the bodies: $m_1, m_2, \qquad M := m_1 + m_2, \qquad \mu := \dfrac{m_1 m_2}{M},$

  $$\nu := \frac{\mu}{M} = \frac{m_1 m_2}{(m_1+m_2)^2}, \quad 0 \le \nu \le \frac{1}{4};$$

  dimensionless PN parameter introduced for circular orbits : $\quad x := \dfrac{(GM\dot{\phi})^{2/3}}{c^2}.$

- Binding energy of two-point-mass system in circular orbits:

$$E(x;\nu) = -\frac{\mu c^2 x}{2}\left(1 + e_{1\mathrm{PN}}(\nu)\,x + e_{2\mathrm{PN}}(\nu)\,x^2 + e_{3\mathrm{PN}}(\nu)\,x^3 + \left(e_{4\mathrm{PN}}(\nu) + \frac{448}{15}\nu \ln x\right)x^4 + \mathcal{O}(x^5)\right),$$

$$e_{1\mathrm{PN}}(\nu) = -\frac{3}{4} - \frac{1}{12}\nu, \qquad e_{2\mathrm{PN}}(\nu) = -\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2,$$

$$e_{3\mathrm{PN}}(\nu) = -\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3,$$

$$e_{4\mathrm{PN}}(\nu) = -\frac{3969}{128} + \left(-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}(2\ln 2 + \gamma_{\mathrm{E}})\right)\nu + \left(-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4$$

($\gamma_{\mathrm{E}}$ is the Euler's constant).

# 4.5PN-Accurate Gravitational-Wave Luminosity for Circular Orbits

$$\mathcal{L}(x;\nu) = \frac{32c^5}{5G}\,\nu^2 x^5 \Big\{ 1 + \ell_{1\mathrm{PN}}(\nu)\,x + 4\pi\,x^{3/2} + \ell_{2\mathrm{PN}}(\nu)\,x^2 + \ell_{2.5\mathrm{PN}}(\nu)\,x^{5/2} + \Big(\ell_{3\mathrm{PN}}(\nu) - \frac{856}{105}\ln(16x)\Big)\,x^3$$

$$+ \ell_{3.5\mathrm{PN}}(\nu)\,x^{7/2} + \Big(\ell_{4\mathrm{PN}}(\nu) + \Big(\frac{232597}{8820} + \frac{20739}{245}\nu\Big)\ln x\Big)\,x^4$$

$$+ \Big(\ell_{4.5\mathrm{PN}}(\nu) - \frac{3424}{105}\pi\ln(16\,x)\Big)\,x^{9/2} + \mathcal{O}\left(x^5\right)\Big\},$$

$$\ell_{1\mathrm{PN}}(\nu) = -\frac{1247}{336} - \frac{35}{12}\nu, \qquad \ell_{2\mathrm{PN}}(\nu) = -\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2, \qquad \ell_{2.5\mathrm{PN}}(\nu) = \Big(-\frac{8191}{672} - \frac{535}{24}\nu\Big)\,\pi,$$

$$\ell_{3\mathrm{PN}}(\nu) = \frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} + \Big(-\frac{134543}{7776} + \frac{41}{48}\pi^2\Big)\,\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3,$$

$$\ell_{3.5\mathrm{PN}}(\nu) = \Big(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\Big)\,\pi,$$

$$\ell_{4\mathrm{PN}}(\nu) = -\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\mathrm{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3$$

$$+\Big(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\mathrm{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3\Big)\nu$$

$$+\Big(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\Big)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4,$$

$$\ell_{4.5\mathrm{PN}}(\nu) = \Big(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} + \Big(\frac{2062241}{22176} + \frac{41}{12}\pi^2\Big)\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\Big)\,\pi.$$

## Post-Newtonian Two-Body Problem (Without Spins and Tidal Effects)

There are two sub-problems, usually analyzed separately:

- problem of deriving equations of motion (EOM),
- problem of computing gravitational-wave luminosities.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **EOM** | $\boxed{\text{N}}$ | $\boxed{\text{1PN}}$ | $\boxed{\text{2PN}}$ | $\boxed{\text{2.5PN}}$ | $\boxed{\text{3PN}}$ | $\boxed{\text{3.5PN}}$ | $\boxed{\text{4PN}}$ | $\boxed{\text{4.5PN}}$ | $\boxed{\text{5PN}^*}$ | $\boxed{\text{5.5PN}}$ | $\boxed{\text{6PN}^{**}}$ | **6.5PN** | **7PN** |
| **Luminosity** | — | — | — | $\boxed{\text{N}}$ | — | $\boxed{\text{1PN}}$ | $\boxed{\text{1.5PN}}$ | $\boxed{\text{2PN}}$ | $\boxed{\text{2.5PN}}$ | $\boxed{\text{3PN}}$ | $\boxed{\text{3.5PN}}$ | $\boxed{\text{4PN}}$ | $\boxed{\text{4.5PN}}$ |

Red color = worked out completely;
orange color = worked out almost completely (as far as I know).

*3 numerical coefficients of the EOB representation of the 5PN dynamics are still controversial.
**4 numerical coefficients of the EOB representation of the 6PN dynamics are unknown.
(EOB = effective one-body)

EOM at orders N, 1PN, 2PN, and 3PN are purely conservative,
EOM at orders 2.5PN and 3.5PN are purely dissipative.

## 2.5PN-Accurate Two-Body Equations of Motion (Without Spins)

- 0PN (Newtonian): Newton 1687.
- 1PN ($\propto v^2/c^2$): Lorentz & Droste 1917 (extended-body derivations);
  Einstein, Infeld, & Hoffmann 1938 (surface-integral method),
  Robertson 1938 (1PN periastron advance);
  Fock 1939 & Petrova 1940 (PhD thesis, published 1949 only)
  (extended-body derivations).
- 2PN ($\propto v^4/c^4$):
  Ohta, Okamura, Kimura, & Hiida 1974 (incomplete),
  Damour & Deruelle 1981 (not complete), Damour 1982,
  Damour & Schäfer 1985 (ADM-gauge Lagrangian),
  Kopeikin 1985 (extended-body derivation),
  Damour & Schäfer 1987 (2PN periastron advance via ADM Hamiltonian).
- 2.5PN ($\propto v^5/c^5$):
  Damour & Deruelle 1981 (not complete), Damour 1982,
  Grishchuk & Kopeikin 1983 (extended-body derivation),
  Schäfer 1985 (ADM-Hamiltonian-based derivation).

- Damour 1983—deriving the rate of the decay $\dot{P}$ of the orbital period of the two-body system directly from 2.5PN-accurate EOM;
  this work ended—at least to a large extent—the quadrupole formula controversy (which was vivid in the 70th/80th of the XX century).
  [See, e.g., D. Kennefick, *Traveling at the Speed of Thought*, Ch. 11.]

## 3PN Conservative Two-Body EOM (Without Spins): the First Four Independent and Mutually Compatible Derivations

The three derivations used $\delta$-sources and dimensional regularization (DR):

- Damour, Jaranowski, & Schäfer (1998–2001),
  ADM-Hamiltonian-based derivation,
  initial purely 3-dim. derivation plagued by two UV-divergence-related ambiguity parameters (one parameter fixed by the requirement of Poincaré invariance),
  final 2001 non-ambiguous derivation used DR;
- Blanchet, Damour, Esposito-Farèse, & Faye (2000–2004),
  harmonic-coordinate-based derivation,
  initial purely 3-dimensional Lorentz-invariant derivation plagued by one UV-divergence-related ambiguity parameter,
  final 2004 non-ambiguous derivation used DR;
- Foffa & Sturani (2011),
  effective-field-theory approach, non-ambiguous derivation using DR.

There exists only one pure 3-dimensional derivation using an extended body model together with the strong-field point-particle limit and a surface-integral approach (in harmonic coordinates):

- Itoh & Futamase (2003–2004).

## 4PN Conservative Two-Body EOM (Without Spins): the First Four Independent and Mutually Compatible Derivations

All four derivations used $\delta$-sources and DR:

- Damour, Jaranowski, & Schäfer (2012–2015),
  ADM-Hamiltonian-based derivation,
  final 2014 derivation used DR and beyond-near-zone information
  taken from Bini & Damour (2013);
- Bernard, Blanchet, Bohé, Faye, Marchand, & Marsat (2016–2017),
  harmonic-coordinate-based derivation,
  final 2017 derivation used DR and beyond-near-zone information
  taken from Bini & Damour (2013);
  then a new treating of IR divergences by means of DR by
  Bernard, Blanchet, Bohé, Faye, Marchand, & Marsat (2017–2018);
- Foffa, Porto, Rothstein, & Sturani (2019),
  effective-field-theory approach,
  non-ambiguous derivation using DR;
- Blümlein, Maier, Marquard, & Schäfer (2020),
  effective-field-theory approach,
  non-ambiguous derivation using DR.

## Gravitational-Wave Energy Flux from Binary System (Without Spins)

- Lowest order (quadrupole formula):
  Einstein 1918, Peters–Mathews 1963, Schäfer 1985.
- 1PN correction ($\mathcal{O}(v^2/c^2)$):
  Wagoner–Will 1976, Blanchet–Schäfer 1989, Jaranowski-Schäfer 1997.
- 1.5PN correction ($\mathcal{O}(v^3/c^3)$):
  Blanchet–Damour 1992, Wiseman 1993.
- 2PN correction ($\mathcal{O}(v^4/c^4)$):
  Blanchet–Damour–Iyer 1995, Will–Wiseman 1995.
- 2.5PN correction ($\mathcal{O}(v^5/c^5)$):
  Blanchet 1996.
- 3PN correction ($\mathcal{O}(v^6/c^6)$):
  Blanchet–Damour–Esposito-Farèse–Iyer 2004.
- 3.5PN correction ($\mathcal{O}(v^7/c^7)$):
  Blanchet 1998.
- 4PN/4.5PN corrections ($\mathcal{O}(v^8/c^8) + \mathcal{O}(v^9/c^9)$):
  Blanchet–Faye–Henry–Larrouturou–Trestini 2023.

## Availability of Balance Equations

$$\boxed{(2001) \quad \frac{\mathrm{d}E_{3.5\mathrm{PN}}}{\mathrm{d}t}} = \boxed{-\mathcal{L}_{3.5\mathrm{PN}} \quad (2004, \text{ only circular orbits})},$$

$$\boxed{(2014) \quad \frac{\mathrm{d}E_{4.5\mathrm{PN}}}{\mathrm{d}t}} = \boxed{-\mathcal{L}_{4.5\mathrm{PN}} \quad (2023, \text{ only circular orbits})}.$$

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